Elements of Geometry

Hull

ELEMENTS OF GEOMETRY,

INCLUDING

PLANE, SOLID, AND SPHERICAL GEOMETRY.

BY

GEO. W. HULL, M. A., Ph. D.,

Professor of Mathematics in the First Pennsylvania State Normal School, Millersville, Pa.; Author of Hull's Arithmetics and Algebra.

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Hull's Geom.

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PREFACE.

In the preparation of this work the author has kept constantly in view not only the object to be attained by the study of Geometry, which is the power of deductive reasoning, but also the needs of the student in the acquisition of this power.

A careful study of the work in the class-room indicates that, if symbols are used too extensively, the language and frequently the logic of the successive steps of a demonstration are impaired; and, on the other hand, if no symbols are used, the subject becomes tedious and the student is discouraged. The endeavor here has been to avoid both of these errors, by the present form and arrangement of the demonstrations.

A student beginning the study of Geometry requires a great deal of aid before he can give a clear and logical demonstration; therefore, at first, the reason for each step of the demonstration is given in small type immediately below the statement. But too much aid of this kind has been found injurious, and leads to careless habits of study. These references have therefore been limited to the first nine theorems of Book I. In all other cases the numbers have been given to indicate the references.

No student can make any great progress in Geometry without frequent practice in original demonstrations; hence a large number of well-selected and well-graded theorems for original thought are given in each book, and, further to obviate the discouraging difficulties which a young student generally encounters in his endeavor to prove a new truth, the diagrams have been given for a few of these theorems, and suggestive references for many others.

Modern mathematical thought is almost unanimously of the opinion that the *Theory of Limits* is the only rational basis of Geometry. Therefore this treatise is based upon this important principle.

Attention is called to the form and arrangement of the demonstrations, and to their brevity and simplicity, due to the use of Continued Proportion. Other commendable features are the concise discussion of the Theory of Limits; the practical solution of the π Proposition; the use of a single letter to represent an angle, a line, or a polygon; the diagrams of Solid and Spherical Geometry; and the disposition of the corollaries and converse propositions.

The author returns his sincere thanks for the assistance which he has received from experienced teachers.

MILLERSVILLE, PA.,
Jan. 1, 1898.

GEORGE W. HULL.

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SUPPLEMENT TO SOLID GEOMETRY.

SUGGESTIONS TO TEACHERS.

THE attention of the teacher is called to the following suggestions:

1. The pupil should be required to fix the fundamental definitions of geometry firmly in mind, and to illustrate them so fully that there can be no doubt of his clearly comprehending them.

2. All definitions, axioms, theorems, corollaries, etc., should be memorized and repeated until they can be given without conscious effort. Demonstrations should not be memorized. The pupil should be taught to think clearly and logically, and to express the thought in concise and elegant language.

3. The construction of the figure is an important part of the demonstration, and no assistance should be given the pupil. The figures should all be constructed with a *ruler* and *compasses.*

4. In going over each book for the first time the "Exercises" can be omitted, and taken in connection with the review.

5. Students should be taught the *Sequence of Propositions.* Thus, the area of a triangle depends upon the area of a parallelogram; the area of a parallelogram depends upon the area of a rectangle, etc. This can be used in any book with excellent results.

6. Students are sometimes at a loss to know in what form written exercises in geometry should be given. We therefore suggest that in all written exercises care be taken to arrange the work to secure clearness of thought and brevity of expression. Symbols and abbreviations can be used with great profit. It is well to begin each statement on a separate line, giving the

reason for each step briefly in parentheses immediately below the line.

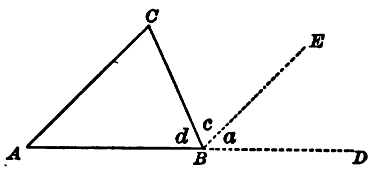
We recommend two forms, the first for *Beginners, Reviews,* or *Examinations,* and the second for *Advanced Pupils.*

Advanced pupils should be required to write on the blackboard only those proportions and equations which are necessary for a clear and rapid demonstration. References, constructions, etc., should be given orally.

1. For Beginners, Reviews, or Examinations.

PROPOSITION XVII. BOOK I.

The sum of the angles of a triangle is equal to two right angles.



Given—The $\triangle ABC$.

To Prove—$\angle A + \angle d + \angle C =$ two right angles.

Dem.—Draw $BE \parallel AC$, and produce AB to D.

Then $\angle A = \angle a,$

(Being ext.-int. ang.)

And $\angle C = \angle c.$

(Being alt.-int. ang.)

But $\angle a + \angle d + \angle c = 2$ rt. $\angle s$.

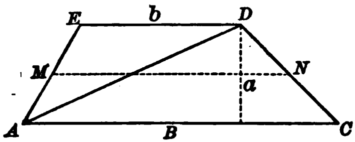
(The sum of all the angles about a point on the same side of a straight line equals two rt. $\angle s$.)

Substituting $\angle A$ for $\angle a$, and $\angle C$ for $\angle c$, we have

$\angle A + \angle d + \angle C =$ two rt. $\angle s$. Q. E. D.

PROPOSITION VII. BOOK IV.

The area of a trapezoid is equal to one half the sum of its bases multiplied by its altitude.



Given—$ACDE$ a trapezoid, whose altitude is a, and upper and lower bases b and B respectively.

To Prove—The area $ACDE = \frac{1}{2}(B + b)a$.

Dem.—Draw the diagonal AD.

Then the area of $\triangle ACD = \frac{1}{2}B \times a$. (1)

(The area of a $\triangle$ is equal to one-half the product of its base and altitude.)

The area of the $\triangle AED = \frac{1}{2}b \times a$. (2)

(For the same reason.)

Adding (1) and (2), the area $ACDE = \frac{1}{2}(B + b)a$. Q. E. D.

2. For Advanced Pupils.

PROPOSITION X. BOOK IV.

In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.

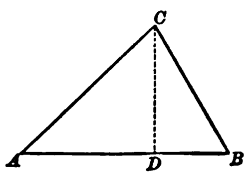


FIG. 1.

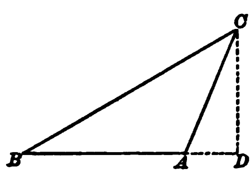


FIG. 2.

Given—B an acute angle of the triangle ABC, with CD perpendicular to AB.

To Prove—$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times DB$.

Dem.—In Fig. 1, $AD = AB - DB$.

In Fig. 2, $AD = BD - AB$.

In the right $\triangle ADC$,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2.$$

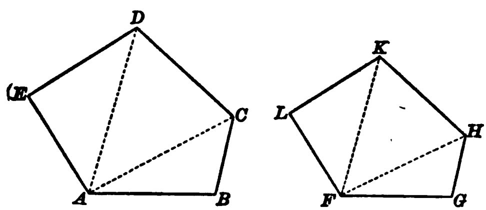
But $\overline{AD}^2 = \overline{AB}^2 + \overline{DB}^2 - 2AB \times DB$,

And $\overline{DC}^2 = \overline{BC}^2 - \overline{DB}^2$.

Adding, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times DB$. Q. E. D.

PROPOSITION XXIX. BOOK IV.

Two similar polygons are to each other as the squares of any two homologous sides.



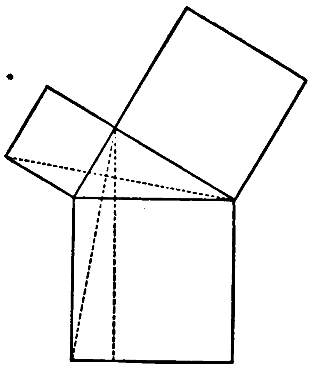
Given—P and P' the areas of the similar polygons EB and LG respectively.

To Prove— $\dfrac{P}{P'} = \dfrac{\overline{AB}^2}{\overline{FG}^2}$.

Dem.—Since $\dfrac{\triangle ABC}{\triangle FGH} = \left(\dfrac{\overline{AC}^2}{\overline{FH}^2}\right) = \dfrac{\triangle ACD}{\triangle FHK} = \left(\dfrac{\overline{AD}^2}{\overline{FK}^2}\right) = \dfrac{\triangle ADE}{\triangle FKL}$,

Then $\dfrac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle FGH + \triangle FHK + \triangle FKL} = \left(\dfrac{\triangle ABC}{\triangle FGH}\right) = \dfrac{\overline{AB}^2}{\overline{FG}^2}$.

Hence $\dfrac{P}{P'} = \dfrac{\overline{AB}^2}{\overline{FG}^2}$. Q. E. D.

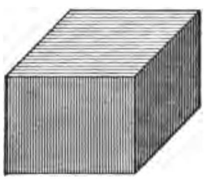


PLANE AND SOLID GEOMETRY.

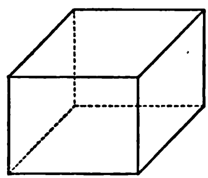
INTRODUCTION.

DEFINITIONS OF TERMS.

1. EVERY material object occupies a definite portion of space. This definite portion of space, considered apart from the body which filled it, is a *geometrical solid.* The material body which filled the space is a *physical solid.*



A physical solid.



A geometrical solid.

A geometrical solid is therefore merely the form of a physical solid. Since geometry deals only with geometrical forms, the word *solid* in this work is used for geometrical solid.

A **Solid** is a definite portion of space, and has *length*, *breadth*, and *thickness.*

2. The limitations of a solid are surfaces, the limitations of a surface are lines, and the limitations of a line are points.

A **Surface** is that which has *length* and *breadth,* without *thickness.*

A **Line** is that which has *length*, but neither *breadth* nor *thickness.*

A **Point** is that which has *position*, but neither *length*, *breadth*, nor *thickness*.

3. Lines.

A *Straight Line* is a line which *never* changes its direction; as *AB*.

A *Curved Line* is a line which changes its direction at *every* point; as *CD*.

A *Broken Line* is a line composed of straight lines; as *EFGH*.

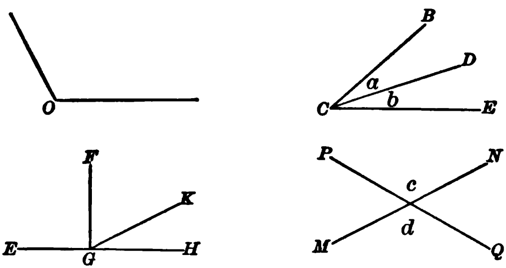
A straight line. A curved line. A broken line.

A straight line is also called a right line. The word *line* is used to indicate a *straight line;* the word *curve* to indicate a *curved line.* A line composed of *straight* and *curved* lines is called a *mixed line.*

4. Angles.

An *Angle* is the *amount of divergence* of two lines which meet at a common point.

The point in which the two lines meet is called the *vertex* of the angle, and the two lines the *sides* of the angle.



If there is but one angle at the same vertex, it may be designated by a single letter, as "the angle *O*." But if two

or more angles have the same vertex, they may be designated by placing a letter in each angle, as "the angle *a*," or by naming the letters at the extremities of the sides, placing the letter at the vertex between the other two, as "the angle *BCD*."

Two angles, such as *BCD* and *DCE*, which have the same vertex and a common side between them, are called *Adjacent* angles.

Two angles are called *vertical*, or *opposite*, when they have the same vertex and their sides extend in opposite directions, as angle *c* and angle *d*.

Equal Angles are angles which have the same divergence, or when applied to each other their sides will extend in the same direction.

When one straight line meets another straight line, making the two adjacent angles equal, each of these angles is called a *Right Angle*, as *EGF* and *FGH*, and the first line is *Perpendicular* to the second.

An *Acute* angle is less than a right angle; as the angle *KGH*.

An *Obtuse* angle is greater than a right angle; as the angle *EGK*.

Intersecting lines not perpendicular to each other are called *Oblique Lines.*

When the sum of two angles is equal to a right angle, each is called the *Complement* of the other; thus, *HGK* is the complement of *KGF*.

When the sum of two angles is equal to two right angles, each is the *Supplement* of the other; thus, *HGK* is the supplement of *KGE*.

It is evident that—

(*a*) *The complements of equal angles are equal.*

(*b*) *The supplements of equal angles are equal.*

5. Surfaces.

A *Plane Surface*, or simply a *Plane*, is a surface such that if any two of its points be connected by a straight line, this line will lie wholly in the surface.

A *Curved Surface* is a surface no portion of which, however small, is plane.

6. Figures.

A *Geometrical Figure* is any combination of points, lines, surfaces, or solids.

A *Plane Figure* is a plane bounded by lines either straight or curved.

Rectilinear Figures are figures bounded by straight lines; *Curvilinear Figures* are figures bounded by curved lines; *Mixtilinear Figures* are figures bounded by straight and curved lines.

7. General Definitions.

Geometry is the science of extension, and treats of the *forms* and *relations* of geometrical magnitudes.

A *Theorem* is a truth requiring demonstration.

A *Lemma* is an auxiliary theorem.

An *Axiom* is a self-evident truth.

A *Problem* is a question to be solved.

A *Postulate* is a problem whose solution is self-evident.

A *Proposition* is a general term applied to theorems, problems, axioms, or postulates.

A *Corollary* is a truth easily drawn from a proposition.

A *Scholium* is a remark upon one or more propositions.

The *Hypothesis* is that which is assumed in the statement of a theorem.

The *Conclusion* is that which follows from the hypothesis.

A proposition is the *Converse* of another when the hy-

pothesis and conclusion of the first are respectively the conclusion and hypothesis of the second.

8. Axioms.

1. *Things which are equal to the same thing, or equal things, are equal to each other.*

2. *If the same operation be performed on equals, the results will be equal.*

3. *The whole is greater than any of its parts.*

4. *The whole is equal to the sum of all its parts.*

5. *A straight line is the shortest distance between two points.*

6. *Only one straight line can be drawn connecting two given points.*

7. *Through the same point, only one straight line can be drawn parallel to a given line.*

9. Postulates.

1. *A straight line can be drawn from one point to another.*

2. *A straight line can be produced to any length.*

10. Symbols of Operation.

$+$, plus.	$\div$, divided by.
$-$, minus.	A^2, exponent.
$\times$, multiplied by.	$\surd$, radical sign.

11. Symbols of Relation.

$=$, equals.	$>$, is greater than.
$\Leftrightarrow$, is equivalent to.	$<$, is less than.
$:$, ratio.	

12. Symbols of Abbreviation.

$\angle$, angle.	$\square$, parallelogram.
$\parallel$, parallel.	$\square$, rectangle.
$\perp$, perpendicular.	$\bigcirc$, circle.
$\triangle$, triangle.	$\therefore$, therefore.

The plural of these terms may be denoted by writing an *s* within or without the symbol; as $\angle s$, $\|s$, etc.

13. Abbreviations of Terms.

Def.,	definition.	Hyp.,	hypothesis.
Ax.,	axiom.	Cons.,	construction.
Post.,	postulate.	Adj.,	adjacent.
Th.,	theorem.	Ext.,	exterior.
Cor.,	corollary.	Int.,	interior.
Alt.,	alternate.	Comp.,	complementary.
Inc.,	included.	Sup.,	supplementary.
Ex.,	exercise.	Dem.,	demonstration.

Q. E. D., *quod erat demonstrandum* (= which was to be proved).

Q. E. F., *quod erat faciendum* (= which was to be done).

PLANE GEOMETRY.

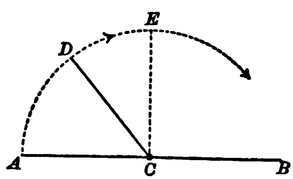
BOOK I.

LINES, ANGLES, AND POLYGONS.

PERPENDICULARS AND ANGLES.

PROPOSITION I. THEOREM.

14. ***From a given point in a straight line only one perpendicular can be erected to that line.***



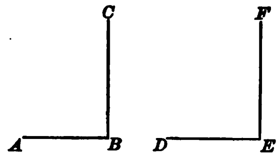
Given—C any point in the straight line AB.

To Prove—That but one perpendicular can be drawn to AB at C.

Dem.—Draw DC, making $\angle DCB > \angle DCA$, and revolve DC about C toward the position of CB.

$\angle DCA$ will constantly increase, and $\angle DCB$ will constantly decrease; hence there can be one position of DC, and but one, where the angles are equal.

Let CE be this position; then CE is the only perpendicular that can be drawn to AB at the point C. Q. E. D.



15. Cor.—*All right angles are equal.*

Given—ABC and DEF right angles.

To Prove—$\angle B = \angle E$.

Dem.—Apply DEF to ABC, placing E on B, and making DE fall on AB.

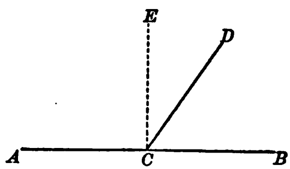
Then EF will coincide with BC; for otherwise we should have two perpendiculars to AB at B, which is impossible.

[*From a given point in a straight line only one perpendicular can be erected to that line.*] § 14

Note.—The method of demonstration employed in this corollary is called *Superposition.*

Proposition II. Theorem.

16. *If a straight line meets another straight line, the sum of the two adjacent angles is equal to two right angles.*



Given—DC meeting AB at C.

To Prove—$\angle ACD + \angle BCD =$ two right angles.

Dem.—Draw CE perpendicular to AB at C.

Then $\angle ACD =$ rt. angle $ACE + \angle DCE$,

[*The whole is equal to the sum of all its parts.*] Ax. 4

And $\angle BCD =$ rt. angle $BCE - \angle DCE$. Ax. 4

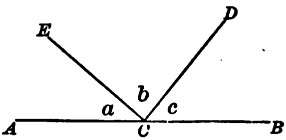
Adding these two equalities, we have

$\angle ACD + \angle BCD =$ two right angles. Ax. 2

[*If the same operation is performed on equals, the results will be equal.*] Q. E. D.

17. COR. 1.—*If one of the two adjacent angles, ACD or DCB, is a right angle, the other is also a right angle.*

18. COR. 2.—*The sum of all the angles formed on the same side of a straight line at a given point is equal to two right angles.*



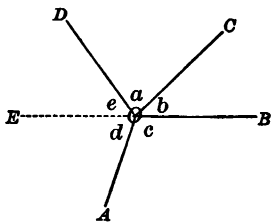
For $\angle a + \angle b = \angle ACD.$

[*The whole is equal to the sum of all its parts.*] Ax. 4.

But $\angle ACD + \angle DCB =$ two right angles.

[*Being sup. adj. $\angle$s.*] § 16

19. COR. 3.—*The sum of all the angles formed about a point in a plane is equal to four right angles.*



Produce *BO* to *E*.

Then $\angle a + \angle b + \angle e =$ two right angles, § 18

And $\angle c + \angle d =$ two right angles. § 18

Adding these two equalities, we have

$$\angle a + \angle b + \angle c + \angle AOD = \text{four right angles.}$$

[*If the same operation is performed on equals, the results will be equal.*] Ax. 2.

EXERCISES.

1. If one of two supplementary adjacent angles is 70°, what is the other angle?

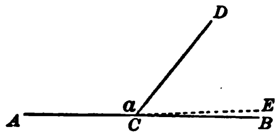
2. How many degrees in each of three angles formed on the same side of a straight line at the same point, if they are equal?

3. How many degrees in each of five angles formed about a point in a plane, if they are equal?

4. In § 18, if $\angle a = 38°$ and $\angle b = 60°$, how many degrees in $\angle c$?

PROPOSITION III. THEOREM.

20. CONVERSELY—***If the sum of two adjacent angles is equal to two right angles, their exterior sides form one and the same straight line.***



Given—$\angle a + \angle DCB =$ two right angles.

To Prove—AC and CB a straight line.

Dem.—If ACB is not a straight line, let ACE be a straight line.

Then $\angle a + \angle DCE =$ two right angles.

[*Being sup. adj.* $\angle$*s.*] § 16

But $\angle a + \angle DCB =$ two right angles. Hyp.

$\therefore \angle a + \angle DCE = \angle a + \angle DCB.$

[*Things that are equal to the same thing are equal to each other.*] Ax. 1

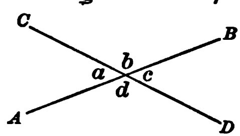
Dropping the common angle a,

$$\angle DCE = \angle DCB.$$

This is impossible unless CE coincides with CB.
Hence ACB is a straight line. Q. E. D.

PROPOSITION IV. THEOREM.

21. ***If two straight lines intersect each other, the opposite or vertical angles are equal.***



Given—The straight line AB intersecting CD.

To Prove— $\angle a = \angle c.$

Dem.—$\angle a + \angle b =$ two right angles.

[*Being sup. adj.* $\angle$*s.*] § 16

For the same reason $\angle b + \angle c =$ two right angles.

Hence $\angle a + \angle b = \angle b + \angle c$. Ax. 1

Dropping the common angle b, Ax. 2

$$\angle a = \angle c.$$

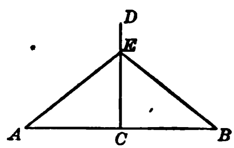
In like manner we may prove $\angle b = \angle d$. Q. E. D.

PROPOSITION V. THEOREM.

22. ***If a perpendicular is erected at the middle point of a straight line, then***

1st. Any point in the perpendicular is equally distant from the extremities of the line.

2d. Any point without the perpendicular is unequally distant from the extremities of the line.



I. **Given**—CD perpendicular to AB at its middle point C.

To Prove—Any point of CD equally distant from A and B.

Dem.—Let E be any point of CD, and draw AE and BE.

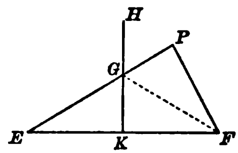
Revolve the figure ACE around CE as an axis, then AC will take the direction of CB, since $\angle ACE = \angle BCE$.

[*Being right angles.*] § 15

And since, by hypothesis, $AC = CB$, the point A will fall on B,

And AE will coincide with BE. Ax. 6

Therefore $AE = BE$. Q. E. D.



II. **Given**—HK perpendicular to EF at its middle point K.

To Prove—Any point without HK unequally distant from E and F.

Dem.—Let P be any point without HK, and draw PE and PF, one of which will cut the perpendicular at some point, as G; from G draw GF.

Then $$PF < FG + GP.$$

[*A straight line is the shortest distance between two points.*] Ax. 5.

But $$FG = EG.$$

[*If a perpendicular is erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.*] § 22

Substituting, $$PF < EG + GP,$$

Or $$PF < PE. \qquad \text{Q. E. D.}$$

In applying the figure ACE to BCE it was found that they coincided throughout.

Hence $$\angle A = \angle B,$$

And $$\angle AEC = \angle BEC.$$

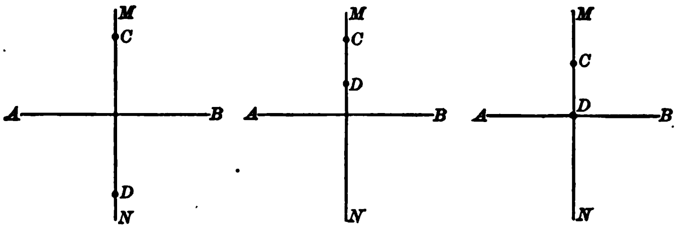
Therefore,

23. Cor. 1.—*If a perpendicular is erected at the middle point of a straight line, and lines be drawn from any point in the perpendicular to the extremities of the line,*

1. *They will make equal angles with the line.*
2. *They will make equal angles with the perpendicular.*

24. Cor. 2.—*The perpendicular at the middle point of a straight line contains all points equally distant from the extremities of the line.*

25. Cor. 3.—*A line which has two points equally distant from the extremities of another line is perpendicular to that line at its middle point.*



Given—The line MN with the two points C and D each equally distant from the extremities of AB.

To Prove—MN perpendicular to AB at its middle point.

Dem.—Suppose a perpendicular to be erected at the middle point of AB, this perpendicular will contain the points C and D.

[*The perpendicular at the middle point of a straight line contains all points equally distant from the extremities of the line.*] § 24

By hypothesis, the line MN contains the points C and D.

Hence MN coincides with the perpendicular at the middle point of AB.

[*Only one straight line can be drawn connecting two given points.*] Ax. 6

EXERCISES.

5. How many degrees are there in the complement of 60°? of 75°? of 120°?

6. How many degrees are there in the supplement of 50°? of 140°? of 210°?

7. What is the complement of $\frac{2}{5}$ of a right angle? of $\frac{5}{7}$ of a right angle? of $\frac{5}{3}$ of a right angle?

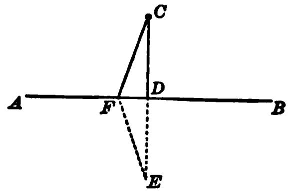
8. What is the supplement of $\frac{4}{5}$ of a right angle? of $\frac{3}{5}$ of a right angle? of $\frac{12}{5}$ of a right angle?

9. How many degrees are there in an angle that is equal to twice its complement?

10. How many degrees are there in an angle that is equal to three times its supplement?

PROPOSITION VI. THEOREM.

26. ***From a given point without a straight line only one perpendicular can be drawn to the line.***



Given—C the point, and CD the perpendicular to AB.

To Prove—That CD is the only perpendicular that can be drawn from C to AB.

Dem.—If possible, suppose CF also perpendicular to AB. Prolong CD to E, making DE equal to CD, and draw FE.

The line FD is perpendicular to CE at its middle point D, hence

$$\angle CFD = \angle EFD.$$

[*If a perpendicular is erected at the middle point of a straight line, and lines be drawn from any point in the perpendicular to the extremities of the line, they will make equal angles with the perpendicular.*] § 23

But, by hypothesis, $\angle CFD$ is a right angle.

Hence its equal, $\angle EFD$, is a right angle,

And CFE is a straight line.

[*If the sum of two adjacent angles is equal to two right angles, their exterior sides form one and the same straight line.*] § 20

This is true only when CF coincides with CD.

[*Only one straight line can be drawn connecting two given points.*] Ax. 6

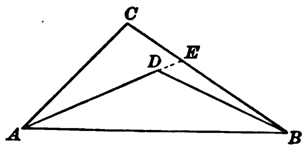
Hence only one perpendicular can be drawn from a given point to a given line. Q. E. D.

EXERCISE.

11. A line drawn perpendicular to the bisector of an angle makes equal angles with its sides.

PROPOSITION VII. THEOREM.

27. ***If two lines are drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but included by them.***



Given—CA and CB two lines drawn from C to the extremities of AB, and DA and DB two other lines similarly drawn, but included by CA and CB.

To Prove— $AC + CB > AD + DB$.

Dem.—Produce AD to E.

Then $AC + CE > AD + DE$.

[*A straight line is the shortest distance between two points.*] Ax. 5

For the same reason,

$$EB + DE > DB.$$

Adding these two inequalities, we have

$$AC + CE + EB + DE > AD + DE + DB.$$

Dropping DE from both members of the inequality, and substituting CB for $CE + EB$, we have

$$AC + CB > AD + DB. \qquad \text{Q. E. D.}$$

EXERCISES.

12. How many degrees are there in an angle the sum of whose complement and supplement is 160°?

13. How many degrees are there in an angle whose supplement is equal to three times its complement?

14. If two straight lines intersect each other making one of the angles 130°, what are each of the other angles?

15. The bisectors of two supplementary adjacent angles are perpendicular to each other.

16. The line which bisects the angle formed by the intersection of two lines bisects also its vertical angle.

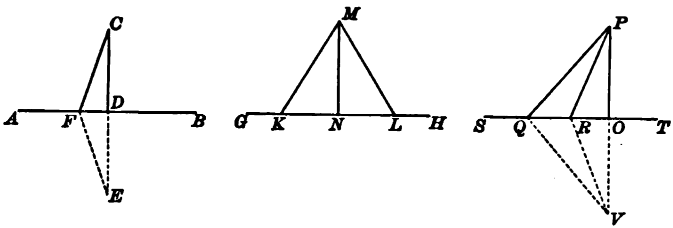
PROPOSITION VIII. THEOREM.

28. ***If from a point without a straight line a perpendicular is let fall to the line, and oblique lines are drawn, then***

1st. The perpendicular is shorter than any oblique line.

2d. Any two oblique lines which cut off on the line equal distances from the foot of the perpendicular are equal.

3d. The oblique line which cuts off on the straight line the greater distance from the foot of the perpendicular is the greater.



I. **Given**—CD perpendicular to AB, and CF any other line drawn from C to AB.

To Prove— $CD < CF$.

Dem.—Produce CD to E, making $DE = CD$, and draw FE.

Then, since FD is perpendicular to CE at its middle point D,

$$CF = FE.$$

[*If a perpendicular is erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.*] § 22

But $CD + DE < CF + FE$.

[*A straight line is the shortest distance between two points.*] Ax. 5

Then $2CD < 2CF$,

Or $CD < CF$. Q. E. D.

II. **Given**—MN perpendicular to GH, and MK and ML oblique lines drawn from M to GH, cutting off equal distances from the foot of the perpendicular.

To Prove— $MK = ML$.

Dem.—Since MN is perpendicular to KL at its middle point N, $MK = ML$.

[*If a perpendicular is erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.*] § 22. Q. E. D.

III. **Given**—PO perpendicular to ST, and PR and PQ oblique lines drawn from P to ST, cutting off unequal distances from the foot of the perpendicular.

To Prove— $PQ > PR$.

Dem.—Produce PO to V, making $OV = PO$, and draw RV and QV.

Then, since SO is perpendicular to PV at its middle point O,

$$PQ = QV, \text{ and } PR = RV.$$

[*If a perpendicular is erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.*] § 22

But $PQ + QV > PR + RV$.

[*If two lines are drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but included by them.*] § 27

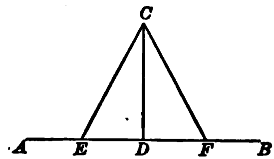
Hence $2PQ > 2PR$,

Or $PQ > PR$. Q. E. D.

Cor.—*Only two equal straight lines can be drawn from a point to a given straight line.*

PROPOSITION IX. THEOREM.

29. CONVERSELY—***If oblique lines be drawn from a point to a line, two equal oblique lines cut off equal distances from the foot of the perpendicular drawn from the point to the line.***



Given—CD perpendicular to AB, and CE and CF equal oblique lines drawn from C to AB.

To Prove— $ED = DF$.

Dem.—If ED were greater than DF, CE would be greater than CF.

[*If from a point without a straight line a perpendicular is let fall to the line, the oblique line which cuts off on the line the greater distance from the foot of the perpendicular is the greater.*] § 28

And if DE were less than DF, CE would be less than CF.

But both of these conclusions are contrary to the hypothesis that $CE = CF$.

Therefore $ED = DF$. Q. E. D.

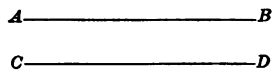
30. COR.—*Of two unequal lines, the greater cuts off the greater distance from the foot of the perpendicular.*

NOTE 1.—The method of demonstration employed in this theorem is called "*Reductio ad Absurdum.*"

NOTE 2.—Thus far the proof for each part of the demonstration has been given in full, immediately after the statement, and printed in *italic* type. In the remainder of this work only the number of the section will be given, but the pupil should always be required to give the full statement.

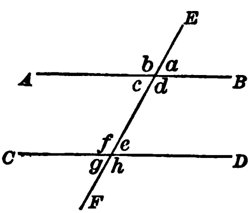
PARALLEL LINES.

31. Parallel Lines are lines which lie in the same plane, but cannot meet however far they are produced; as *AB* and *CD*.



It follows from this definition that straight lines which are not parallel, lying in the same plane, will meet one another when sufficiently produced.

32. When two straight lines, *AB* and *CD*, are cut by a third line *EF*, called a *transversal*, or *secant line*, then



1. The four angles, *a*, *b*, *g*, and *h*, without the two lines, are called *Exterior* angles.
2. The four angles, *c*, *d*, *e*, and *f*, within the two lines, are called *Interior* angles.
3. The angles *d* and *f*, or *c* and *e*, are called *Alternate-interior* angles.
4. The angles *b* and *h*, or *a* and *g*, are called *Alternate-exterior* angles.
5. The pairs of angles *a* and *e*, *b* and *f*, *d* and *h*, or *c* and *g*, are called *Exterior-interior* angles.

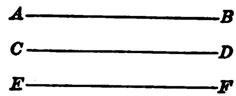
When several lines are drawn through the same point, they are said to have different directions; as *bA*, *bE*, and *aB*.

Parallel lines have the *same direction* when they lie on the same side of the transversal, or when they lie on the same side of a straight line joining their corresponding extremities; as *aB* and *eD*.

Parallel lines have *opposite directions* when they lie on opposite sides of the transversal, or when they lie on opposite sides of a straight line joining their corresponding extremities; as *bA* and *eD*.

PROPOSITION X. THEOREM.

33. ***Two straight lines parallel to the same straight line are parallel to each other.***



Given—AB and CD parallel to EF.

To Prove—AB parallel to CD.

Dem.—If AB and CD are not parallel, they will meet in some point if sufficiently produced. § 31

We should then have two lines drawn through the same point parallel to the same line EF, which is impossible. Ax. 7

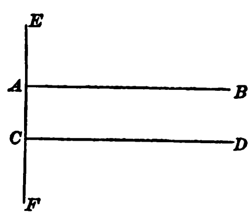
Therefore AB and CD cannot meet, and are parallel.

Q. E. D.

34. COR.—*A line parallel to one of two parallel lines is parallel to the other also.*

PROPOSITION XI. THEOREM.

35. ***Two straight lines in the same plane perpendicular to the same straight line are parallel.***



Given—AB and CD perpendicular to EF.

To Prove—AB and CD parallel.

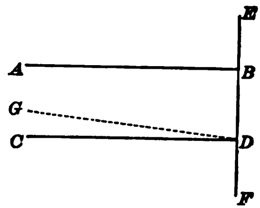
Dem.—If AB and CD are not parallel, they will meet in some point if sufficiently produced. § 31

We should then have two lines drawn from the same point perpendicular to the same line EF, which is impossible. § 26

Therefore AB and CD cannot meet, and are parallel.

Q. E. D.

36. COR.—*A straight line perpendicular to one of two parallels is perpendicular to the other also.*



Given—AB and CD parallel, and EF perpendicular to AB.

To Prove—EF perpendicular to CD.

Dem.—If EF is not perpendicular to CD, draw GD perpendicular to EF.

Then AB and GD are parallel. § 35

But, by hypothesis, AB and CD are parallel.

We should then have two lines drawn from the same point D parallel to AB, which is impossible. Ax. 7

Therefore EF is perpendicular to CD.

EXERCISES.

17. If a straight line intersect two parallel lines, making two of the exterior angles 150°,

a. How many degrees are there in each of the other exterior angles?

b. How many degrees are there in each of the interior angles?

c. How many angles are acute? how many are obtuse?

18. Prove that the shortest distance between two parallel lines is the perpendicular which joins them.

19. How can a farmer tell whether the opposite sides of his farm are parallel?

20. If two supplementary adjacent angles are bisected, what angles are complementary? what angles are supplementary?

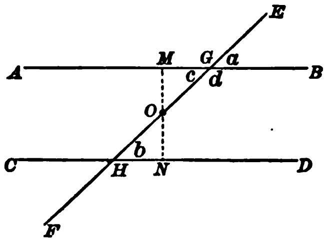
PROPOSITION XII. THEOREM.

37. ***If two parallels are cut by a third line, then***

1. The alternate-interior angles are equal.

2. The exterior-interior angles are equal.

3. The sum of the interior angles on the same side of the transversal is equal to two right angles.



Given—AB and CD parallel and cut by EF.

To Prove, 1st— $\angle c = \angle b$.

Dem.—Through O, the middle point of GH, draw MN perpendicular to AB.

Then MN is also perpendicular to CD. § 36

Apply the figure HON to the figure MOG, placing OH on its equal OG, and the line ON will take the direction of OM, since the angles HON and MOG are equal. § 21

Since H falls on G, and HN and GM are both perpendicular to MN, HN and GM must coincide, or we should have two perpendiculars from the same point G to the same straight line MN, which is impossible. § 26

Hence $\angle c = \angle b$.

To Prove, 2d— $\angle a = \angle b$.

Dem.—Since	$\angle a = \angle c$,	§ 21
And	$\angle b = \angle c$,	1st part, § 37
Then	$\angle a = \angle b$.	Ax. 1

To Prove, 3d— $\angle b + \angle d =$ two right angles.

Dem.—Since $\angle c + \angle d =$ two right angles, § 16

And $\angle c = \angle b$, 1st part, § 37

Then $\angle b + \angle d =$ two right angles. Q. E. D.

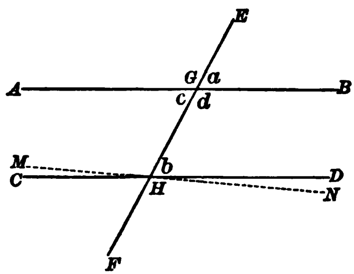
PROPOSITION XIII. THEOREM.

38. *If a straight line intersect two other straight lines, these two lines will be parallel—*

1. When the alternate-interior angles are equal.

2. When the exterior-interior angles are equal.

3. When the sum of the interior angles on the same side of the transversal is equal to two right angles.



Given—AB and CD two straight lines, cut by EF.

To Prove—AB and CD parallel.

1st.—When $\angle b = \angle c$.

Dem.—Through H draw MN parallel to AB.

Since the parallels AB and MN are cut by EF,

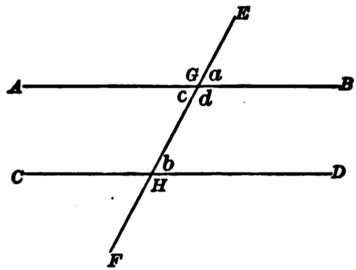
Then $\angle c = \angle GHN$. § 37

But, by hypothesis, $\angle c = \angle b$.

Hence $\angle b = \angle GHN$. Ax. 1

But this is impossible unless CD coincides with MN.

Therefore CD is parallel to AB.



2d.—When $\angle a = \angle b$.

Dem.—Since $\angle a = \angle c$, § 21

And, by hypothesis, $\angle a = \angle b$,

Then $\angle b = \angle c$. Ax. 1

But when $\angle b = \angle c$, AB and CD are parallel. 1st part, § 38

3d.—When $\angle b + \angle d =$ two right angles.

Dem.—Since $\angle c + \angle d =$ two right angles, § 16

And, by hypothesis, $\angle b + \angle d =$ two right angles,

Then $\angle c + \angle d = \angle b + \angle d$. Ax. 1

Dropping the common angle d, Ax. 2

$$\angle c = \angle b.$$

But when $\angle c = \angle b$, AB and CD are parallel.

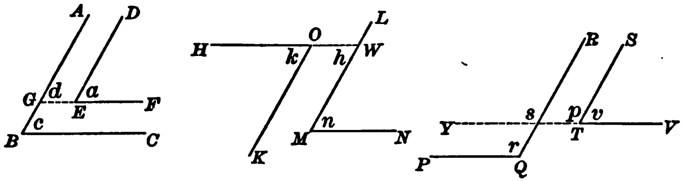
1st part, § 38. Q. E. D.

Proposition XIV. Theorem.

39. *If two angles have their sides respectively parallel,*

1. They are equal, if their sides lie in the same or in opposite directions.

2. They are supplementary, if their sides do not lie in the same or in opposite directions.



I. **Given**—The angles *ABC* and *DEF* with their sides respectively parallel and lying in the same direction.

To Prove— $\angle a = \angle c$.

Dem.—Produce *FE* to *G*.

Then	$\angle a = \angle d$,	§ 37
And	$\angle c = \angle d$.	§ 37
Hence	$\angle a = \angle c$.	Ax. 1

II. **Given**—The angles *HOK* and *LMN* with their sides respectively parallel and lying in opposite directions.

To Prove— $\angle k = \angle n$.

Dem.—Produce *HO* to *W*.

Then	$\angle k = \angle h$,	§ 37
And	$\angle n = \angle h$.	§ 37
Hence	$\angle k = \angle n$.	Ax. 1

III. **Given**—The angles *PQR* and *STV* with their sides respectively parallel and not lying in the same or in opposite directions.

To Prove—Angles r and v supplementary.

Dem.—Produce *VT* to *Y*.

Then	$\angle r = \angle s$,	§ 37
And	$\angle p = \angle s$.	§ 37
Hence	$\angle r = \angle p$.	Ax. 1

But $\angle p$ and $\angle v$ are supplementary. § 16

Substituting $\angle r$ for $\angle p$,

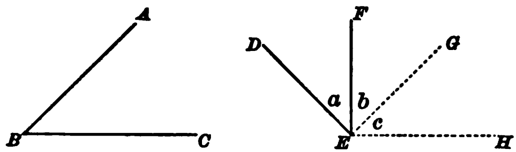
We have $\angle r$ and $\angle v$ supplementary. Ax. 1. Q. E. D.

Proposition XV. Theorem.

40. *If two angles have their sides respectively perpendicular,*

1. They are equal, if both are acute or both obtuse.

2. They are supplementary, if one is acute and the other obtuse.



I. **Given**—AB perpendicular to DE, and CB perpendicular to EF.

To Prove— $\angle B = \angle a$.

Dem.—Construct $\angle c$ with its sides respectively perpendicular to the sides of $\angle a$.

Then the sides of $\angle c$ will be respectively parallel to the sides of $\angle B$, § 35

And $\angle c = \angle B$. § 39

By construction, $\angle a + \angle b =$ one right angle,

And $\angle b + \angle c =$ one right angle.

Hence $\angle a + \angle b = \angle b + \angle c$. Ax. 1

Dropping the common angle b, Ax. 2

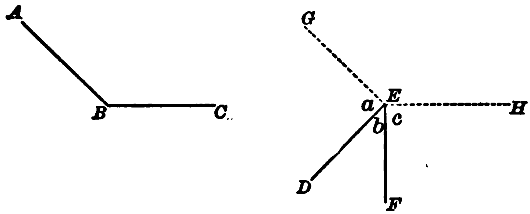
$$\angle a = \angle c.$$

But $\angle B = \angle c$.

Therefore $\angle a = \angle B$. Ax. 1

Note.—This proposition can be readily proved when both angles are obtuse, by producing FE an equal distance below the line EH to F', and regarding DEH and GEF' the obtuse angles.

The pupil should be required to construct the figure and give the demonstration in full.



II. **Given**—AB perpendicular to DE, and CB to EF.

To Prove—$\angle B$ and $\angle b$ supplementary.

Dem.—Construct $\angle GEH$ with its sides respectively perpendicular to the sides of $\angle b$.

Then the sides of $\angle GEH$ will be respectively parallel to the sides of $\angle B$, § 35

And $\angle GEH = \angle B$. § 39

Now, $\angle a + \angle c + \angle b + \angle GEH =$ four right angles. § 19

By construction, $\angle a + \angle c =$ two right angles.

Subtracting, $\angle b + \angle GEH =$ two right angles. Ax. 2

But $\angle GEH = \angle B$.

Substituting, $\angle b + \angle B =$ two right angles, and are therefore supplementary. Q. E. D.

EXERCISES.

21. The sum of the four angles formed by the intersection of two lines is equal to four right angles.

22. If two straight lines are cut by a third line, making the sum of the two interior angles on the same side of the transversal less or greater than two right angles, the lines are not parallel.

23. The shortest line that can be drawn from a point to a given straight line is perpendicular to the line.

24. Two angles are complementary, and the greater exceeds the less by 32°; how many degrees are there in each angle?

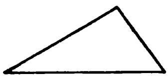
25. Two angles are supplementary, and the greater is five times the less; how many degrees are there in each angle?

TRIANGLES.

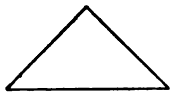
41. A plane **Triangle** is a portion of a plane bounded by three straight lines.

42. Triangles are classified according to their *sides* and according to their *angles.*

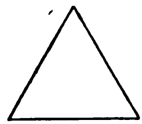
43. By **Sides.**—A triangle is called *Scalene* when no two of its sides are equal; *Isosceles* when two of its sides are equal; and *Equilateral* when all its sides are equal.



Scalene



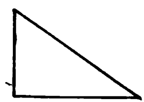
Isosceles



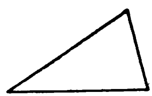
Equilateral

44. By **Angles.**—A triangle is called *Acute* when all its angles are acute; *Obtuse* when one of its angles is obtuse; and *Right* when one of its angles is a right angle.

Obtuse

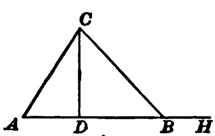


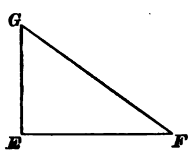
Right

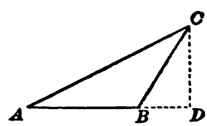


Acute

45. Parts of a Triangle.—The bounding lines, AB, BC, and CA, are called the *sides* of the triangle, their sum,







$AB + BC + CA$, is called the *perimeter*, and their points of intersection, A, B and C, are called the *vertices.*

The *angles* of the triangle are the angles *ABC*, *BCA*, and *CAB*. An *Exterior angle* of a triangle is the angle included between any side and an adjacent side produced; as the angle *CBH*.

In a right triangle the side *FG* opposite the right angle is called the *Hypotenuse*, and the other sides, *EF* and *EG*, are the *legs*.

The *Base* of a triangle is the side upon which it seems to rest. Any side may be taken as the base.

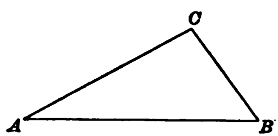
The *Vertical Angle* is the angle opposite the base. The *Altitude* of a triangle is the perpendicular distance from the vertex to the base, or to the base produced; as *CD*.

(a) The equal angles of two triangles are called *Homologous Angles;* and the sides opposite the equal angles are called *Homologous Sides.*

46. The three lines bisecting the angles of a triangle are called the *Bisectors* of the angles of the triangle; and the three lines drawn from the vertices to the middle points of the opposite sides are called the *Medial lines* of the triangle.

PROPOSITION XVI. THEOREM.

47. ***Either side of a triangle is greater than the difference of the other two sides.***



Given—The triangle *ABC*, with *AB* greater than *AC*.

To Prove— $BC > AB - AC.$

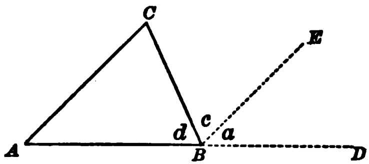
Dem.—We have $BC + AC > AB.$ Ax. 5.

Subtracting *AC* from both members of the inequality,

$BC > AB - AC.$ Q. E. D.

PROPOSITION XVII. THEOREM.

48. ***The sum of the angles of a triangle is equal to two right angles.***



Given—The triangle ABC.

To Prove—$\angle A + \angle d + \angle C =$ two right angles.

Dem.—Draw BE parallel to AC, and produce AB to D.

Since the parallels AC and BE are cut by AD and BC,

$\angle A = \angle a$. § 37

$\angle C = \angle c$. § 37

But $\angle a + \angle d + \angle c =$ two right angles. § 18

Substituting $\angle A$ for $\angle a$, and $\angle C$ for $\angle c$, we have

$\angle A + \angle d + \angle C =$ two right angles. Q. E. D.

49. COR. 1.—*An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

50. COR. 2.—*An exterior angle of a triangle is greater than either of the opposite interior angles.*

51. COR. 3.—*Any angle of a triangle is equal to two right angles less the sum of the other two angles.*

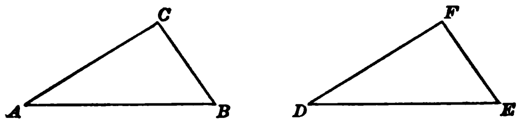
52. COR. 4.—*A triangle cannot have two right angles, nor two obtuse angles.*

53. COR. 5.—*The sum of the two acute angles of a right triangle is equal to one right angle.*

54. COR. 6.—*If two angles of one triangle are respectively equal to two angles of another triangle, the third angles are equal.*

PROPOSITION XVIII. THEOREM.

55. ***Two triangles are equal when two sides and the included angle of one are respectively equal to two sides and the included angle of the other.***



Given—The triangles ABC and DEF, with

$$AB = DE, \; AC = DF, \text{ and } \angle A = \angle D.$$

To Prove— $\triangle ABC = \triangle DEF.$

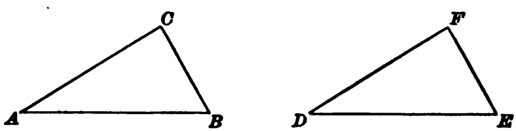
Dem.—Place the triangle ABC upon the triangle DEF, so that AB coincides with its equal DE; and since $\angle A$ equals $\angle D$, the line AC will take the direction of DF; and since AC equals DF, the point C will fall on F,

And BC will coincide with EF. Ax. 6.

Therefore ABC and DEF coincide throughout, and are equal. Q. E. D.

PROPOSITION XIX. THEOREM.

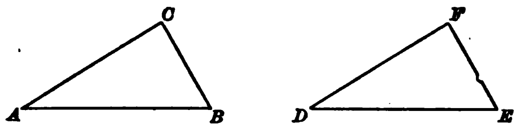
56. ***Two triangles are equal when two angles and the included side of one are respectively equal to two angles and the included side of the other.***



Given—The two triangles ABC and DEF, with

$$AB = DE, \; \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To Prove— $\triangle ABC = \triangle DEF.$



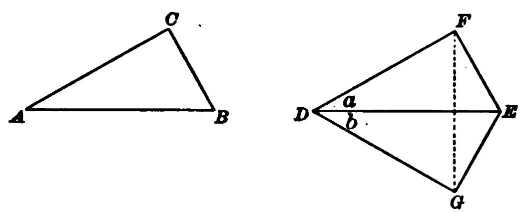
Dem.—Place the triangle ABC upon the triangle DEF, so that AB coincides with its equal DE; and since $\angle A$ equals $\angle D$, the line AC will take the direction of DF, and the point C will fall somewhere on DF or DF produced. Since $\angle B$ equals $\angle E$, the line BC will take the direction of EF, and the point C will fall somewhere on EF or EF produced.

If the point C falls on both DF and EF, it must fall on their intersection F.

Therefore ABC and DEF coincide throughout, and are equal. Q. E. D.

Proposition XX. Theorem.

57. ***Two triangles are equal when three sides of one are respectively equal to three sides of the other.***



Given—The triangles ABC and DEF, with

$$AB = DE,\ AC = DF, \text{ and } BC = EF.$$

To Prove— $\triangle ABC = \triangle DEF$.

Dem.—Place the triangle ABC so that AB coincides with its equal DE, and C falls at G, on the opposite side of DE from F; and draw FG.

Then, since $DF = DG$, and $EF = EG$, the line DE has two points equally distant from the extremities of FG, and is perpendicular to FG at its middle point, § 25

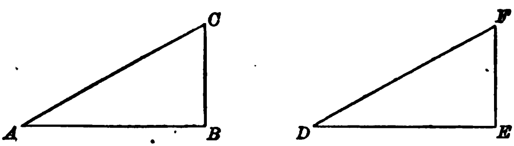
And $\angle a = \angle b$. § 23

Hence $\triangle DEF = \triangle DEG$, § 55

Or $\triangle ABC = \triangle DEF$. Q. E. D.

PROPOSITION XXI. THEOREM.

58. ***Two right triangles are equal when the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.***



Given—The right triangles ABC and DEF, with

$$AC = DF, \text{ and } BC = EF.$$

To Prove— $\triangle ABC = \triangle DEF$.

Dem.—Place the triangle ABC upon the triangle DEF, so that BC coincides with its equal EF; and since the right angle B equals the right angle E (§ 15), the line BA will take the direction of ED, and since AC equals DF, the point A will fall on D. § 29

Therefore ABC and DEF coincide throughout, and are equal. Q. E. D.

59. COR. 1.—*Two right triangles are equal when the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.*

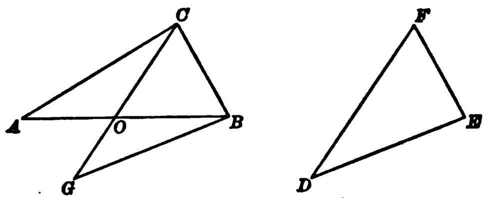
The other acute angles will be equal, by § 54, and the triangles are equal, by § 56.

60. COR. 2.—*Two right triangles are equal when a side and an acute angle of one are respectively equal to a side and an homologous acute angle of the other.*

This is true by § 56.

PROPOSITION XXII. THEOREM.

61. *If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third side is greater in the triangle having the greater included angle.*



Given—The triangles ABC and DEF, with

$$AC = DF, \ BC = EF, \text{ and } \angle C > \angle F.$$

To Prove— $AB > DE$.

Dem.—Place the triangle DEF on the triangle ABC, so that EF falls on its equal BC, and DEF will take the position of GBC.

In the triangle AOC,

$$AO + OC > AC. \qquad \text{Ax. 5}$$

In the triangle GOB,

$$OB + OG > GB. \qquad \text{Ax. 5}$$

Adding these two inequalities, and observing that $AO + OB = AB$, and $OC + OG = GC$, we have

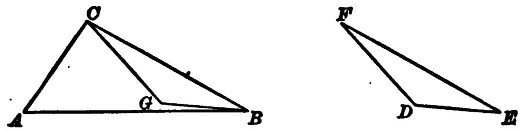
$$AB + GC > AC + GB.$$

Dropping GC and its equal AC, we have

$$AB > GB, \text{ or } AB > DE. \qquad \text{Q. E. D.}$$

SCHOLIUM.—The point G might fall on AB; the theorem would then be true by Ax. 3.

The point G might fall within the triangle. The demonstration would then be as follows:



Given—The triangles ABC and DEF, with

$$AC = DF,\ BC = EF, \text{ and } \angle C > \angle F.$$

To Prove— $AB > DE$.

Dem.—Place the triangle DEF on the triangle ABC, so that EF falls on its equal BC, and DEF will take the position of GBC.

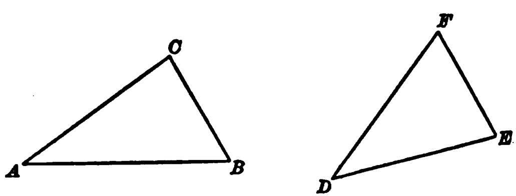
Then $AB + AC > GB + GC$. § 27

Dropping AC and its equal GC, we have

$AB > GB$, or $AB > DE$. Q. E. D.

PROPOSITION XXIII. THEOREM.

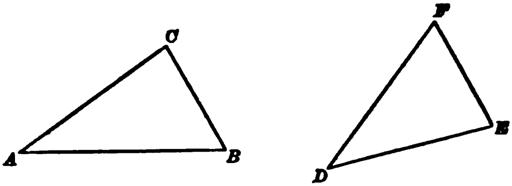
62. CONVERSELY—*If two triangles have two sides of the one respectively equal to two sides of the other, but the third sides unequal, the included angle is greater in the triangle having the greater third side.*



Given—The triangles ABC and DEF, with

$$AC = DF,\ BC = EF, \text{ and } AB > DE.$$

To Prove— $\angle C > \angle F$.



Dem.—If $\angle C$ were equal to $\angle F$, the triangles ABC and DEF would be equal, § 55

And AB would equal DE.

If $\angle C$ were less than $\angle F$,

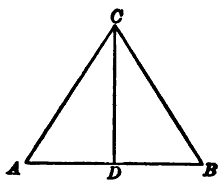
Then AB would be less than DE. § 61

Both of these conclusions are contrary to the hypothesis that AB is greater than DE.

Hence $\angle C > \angle F$. Q. E. D.

PROPOSITION XXIV. THEOREM.

68. ***In an isosceles triangle, the angles opposite the equal sides are equal.***



Given—The triangle ABC, isosceles, with $AC = BC$.

To Prove— $\angle A = \angle B$.

Dem.—Draw CD perpendicular to AB.

Then, in the two right triangles ADC and BDC, CD is common and AC equals BC. Hyp.

Hence the two triangles are equal in all their parts. § 58

Therefore $\angle A = \angle B$. § 45 (*a*). Q. E. D.

64. Cor. 1.—Since the triangles ADC and BDC are equal,

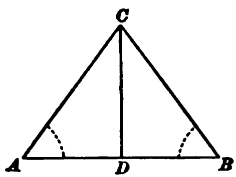
$$AD = DB, \text{ and } \angle ACD = \angle BCD.$$

Hence, *A line drawn from the vertex of an isosceles triangle perpendicular to the base bisects the base and also the vertical angle.*

65. Cor. 2.—*An equilateral triangle is also equiangular.*

Proposition XXV. Theorem.

66. Conversely—***If two angles of a triangle are equal, the sides opposite them are equal.***



Given—The triangle ABC, with $\angle A = \angle B$.

To Prove— $AC = BC$.

Dem.—Draw CD perpendicular to AB.

Then in the two right triangles ADC and BDC, CD is common and $\angle A = \angle B$.

Hence the two triangles are equal in all their parts. § 60

Hence $AC = BC$. § 45 (*a*). Q. E. D.

67. Cor.—*An equiangular triangle is also equilateral.*

EXERCISES.

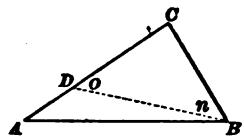
26. A line joining the vertex and the middle of the base of an isosceles triangle bisects the vertical angle, and is perpendicular to the base.

27. A line bisecting the vertical angle of an isosceles triangle is perpendicular to the base at its middle point.

4

PROPOSITION XXVI. THEOREM.

68. ***In any triangle the greater angle lies opposite the greater side.***



Given—The triangle ABC, with $AC > BC$.

To Prove— $\angle B > \angle A$.

Dem.—Lay off $CD = BC$, and draw DB.

The triangle DBC will be isosceles, and

$\angle o = \angle n$. § 63

Since $\angle o$ is an exterior angle of the triangle ABD,

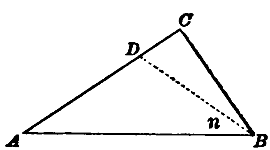
$\angle o > \angle A$. § 50

Now, $\angle B > \angle n$ (Ax. 3), and $\angle n = \angle o$, and $\angle o > A$.

Therefore $\angle B > \angle A$. Q. E. D.

PROPOSITION XXVII. THEOREM.

69. CONVERSELY—*In any triangle the greater side lies opposite the greater angle.*



Given—The triangle ABC, with $\angle B > \angle A$.

To Prove— $AC > BC$.

Dem.—Draw DB, making $\angle n$ equal to $\angle A$.

Then $AD = DB$. § 66

But $DB + DC > BC$. Ax. 5

Substituting AD for its equal DB, we have

$AD + DC$, or $AC > BC$. Q. E. D.

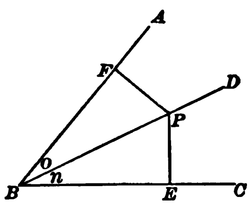
EXERCISES.

28. How many degrees are there in each angle of an equiangular triangle?

29. Any side of a triangle is less than half the sum of the three sides of the triangle.

PROPOSITION XXVIII. THEOREM.

70. ***Any point in the bisector of an angle is equally distant from the sides of the angle; and any point not in the bisector is unequally distant from the sides of the angle.***



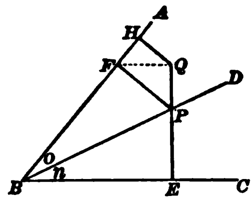
I. **Given**—Any point, as P, in the bisector of the angle ABC.

To Prove—The perpendicular $PE =$ the perpendicular PF.

Dem.—In the right triangles BEP and BFP, BP is common, and $\angle n = \angle o$ by hypothesis.

Then $\triangle BEP = \triangle BFP$. § 59

Hence $PE = PF$. § 45 (*a*)



II. **Given**—Any point as Q not in the bisector of the angle ABC.

To Prove—The perpendicular QH less than the perpendicular QE.

Dem.—Draw QF.

In the right triangle QHF,

$$QH < QF. \qquad \S\ 69$$

But $$QF < QP + PF. \qquad \text{Ax. 5.}$$

Substituting PE for its equal PF, we have

$$QF < QP + PE, \text{ or } QE.$$

Now, $$QH < QF < QE.$$

Hence $$QH < QE. \qquad \text{Q. E. D.}$$

71. SCHOLIUM.—*The bisector of an angle contains all points equally distant from the sides of the angle.*

EXERCISES.

30. If two angles of a triangle are 40° and 60° respectively, what is the third angle?

31. One angle of a right triangle is 26°; what is the third angle?

32. If two angles of a triangle are each 60°, what is the other angle, and what is the kind of triangle?

33. The third angle of an isosceles triangle is four times each angle at the base; how many degrees in each angle?

34. If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.

QUADRILATERALS.

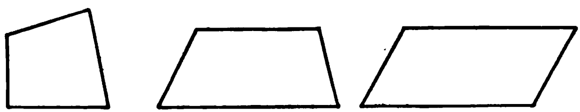
72. A **Quadrilateral** is a plane figure bounded by four straight lines.

73. Quadrilaterals are divided into three classes: the *Trapezium*, the *Trapezoid*, and the *Parallelogram.*

74. A **Trapezium** is a quadrilateral no two of whose sides are parallel.

A **Trapezoid** is a quadrilateral which has only two of its sides parallel.

A **Parallelogram** is a quadrilateral whose opposite sides are parallel.



Trapezium *Trapezoid* *Parallelogram*

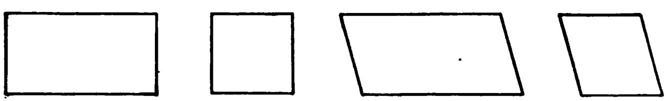
75. Parallelograms are divided into *Rectangles and Rhomboids.*

76. A **Rectangle** is a parallelogram whose angles are right angles.

A **Square** is an equilateral rectangle.

77. A **Rhomboid** is a parallelogram whose angles are oblique.

A **Rhombus** is an equilateral rhomboid.



Rectangle *Square* *Rhomboid* *Rhombus*

78. Parts of a Quadrilateral.—The bounding lines are the *sides* of the quadrilateral; the angles formed by the sides are the *angles* of the quadrilateral.

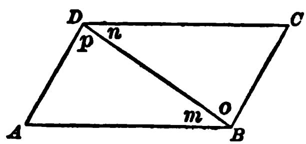
The *Base* of a quadrilateral is the side upon which it seems to rest.

The *bases* of a parallelogram or a trapezoid are the side upon which it seems to stand and its parallel; the *altitude* is the perpendicular distance between them.

The *Diagonal* of a quadrilateral is a straight line joining any two vertices not consecutive.

PROPOSITION XXIX. THEOREM.

79. *The opposite sides and angles of a parallelogram are equal.*



Given—The parallelogram $ABCD$.

To Prove—

$AB = CD$, $AD = BC$, $\angle A = \angle C$, and $\angle B = \angle D$.

Dem.—Draw the diagonal DB.

Since the parallels AB and CD are cut by DB,

$\angle m = \angle n$. § 37

Since the parallels AD and BC are cut by DB,

$\angle o = \angle p$. § 37

Then, in the triangles ABD and BDC, DB is common, and we have two angles and the included side in each respectively equal.

Therefore $\triangle ABD = \triangle BDC$. § 56

Whence $AB = DC$, $AD = BC$, $\angle A = \angle C$, and $\angle B = \angle D$, being homologous parts. Q. E. D.

80. COR. 1.—*A diagonal of a parallelogram divides it into two equal triangles.*

81. COR. 2.—*Parallels included between parallels are equal.*

EXERCISES.

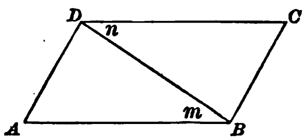
35. If one angle of a parallelogram is 45°, how many degrees are there in each of the other angles?

36. If one base of a trapezoid be extended in both directions, the sum of the exterior angles is equal to the sum of the two opposite interior angles.

37. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

PROPOSITION XXX. THEOREM.

82. CONVERSELY—*If the opposite sides of a quadrilateral are respectively equal, or if the opposite angles are respectively equal, the figure is a parallelogram.*



I. **Given**—The quadrilateral $ABCD$, with

$$AB = DC \text{ and } AD = BC.$$

To Prove—$ABCD$ a parallelogram.

Dem.—Draw the diagonal DB.

Then in the triangles ABD and BCD, DB is common, and, by hypothesis, $AB = DC$ and $AD = BC$.

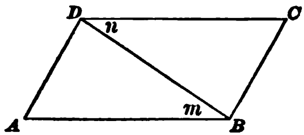
Hence $\triangle ABD = \triangle DBC$, § 57

And $\angle m = \angle n$. § 45 (*a*)

Therefore AB is parallel to DC. § 38

In like manner it may be shown that AD is parallel to BC.

Hence the quadrilateral $ABCD$ has its opposite sides parallel, and is therefore a parallelogram by definition. § 74



II. **Given**—The quadrilateral $ABCD$, with

$$\angle A = \angle C \text{ and } \angle B = \angle D.$$

To Prove—$ABCD$ a parallelogram.

Dem.—The diagonal DB divides the quadrilateral into two triangles the sum of whose angles is equal to the sum of the angles of the quadrilateral.

The sum of the angles of each triangle is two right angles (§ 48); hence the sum of the angles of the quadrilateral is equal to four right angles.

Therefore $\angle A + \angle B + \angle C + \angle D =$ four right angles.

But $\angle A + \angle D = \angle B + \angle C$. Hyp.

Then $\angle A + \angle D =$ two right angles,

And AB is parallel to CD. § 38

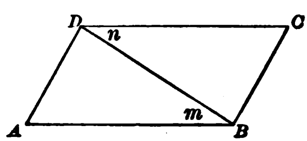
In like manner it may be shown that AD is parallel to BC.

Therefore $ABCD$ is a parallelogram by definition. § 74

Q. E. D.

PROPOSITION XXXI. THEOREM.

83. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given—The quadrilateral $ABCD$, with AB equal and parallel to DC.

To Prove—$ABCD$ a parallelogram.

Dem.—Draw the diagonal BD.

Since the parallels AB and CD are cut by DB,

$\angle m = \angle n$. § 37

Then, in the triangles ABD and BCD, DB is common, $AB = DC$ by hypothesis, and $\angle m = \angle n$.

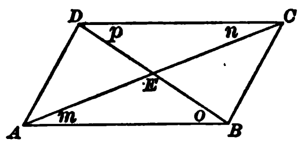
Hence $\triangle ABD = \triangle BCD$, § 55

And $AD = BC$. § 45 (a)

Therefore $ABCD$ is a parallelogram. § 82. Q. E. D.

Proposition XXXII. Theorem.

84. *The diagonals of a parallelogram bisect each other.*



Given—The parallelogram $ABCD$, with the diagonals AC and BD intersecting at E.

To Prove— $AE = EC$, and $BE = ED$.

Dem.—Since AB and CD are parallel,

$\angle m = \angle n$, and $\angle o = \angle p$. § 37

Since $ABCD$ is a parallelogram, $AB = CD$. § 79

Hence $\triangle AEB = \triangle DEC$. § 56

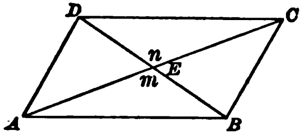
Whence $AE = EC$, and $BE = ED$. § 45 (a). Q. E. D.

85. Cor. 1.—*The diagonals of a rhombus bisect each other at right angles.*

86. Cor. 2.—*The diagonals of a rhombus bisect its opposite angles.*

PROPOSITION XXXIII. THEOREM.

87. CONVERSELY—*If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given—The quadrilateral $ABCD$, whose diagonals AC and BD bisect each other at E.

To Prove—$ABCD$ a parallelogram.

Dem.—In the triangles ABE and DEC,

	$AE = EC$, and $BE = ED$,	Hyp.
And	$\angle m = \angle n$.	§ 21
Hence	$\triangle ABE = \triangle DEC$,	§ 55
And	$AB = DC$.	§ 45 (*a*)
In like manner,	$AD = BC$.	

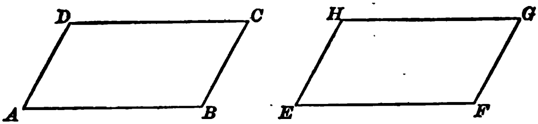
Therefore $ABCD$ is a parallelogram. § 82. Q. E. D.

88. COR. 1.—*If the diagonals of a quadrilateral are equal and bisect each other at right angles, the figure is a square.*

89. COR. 2.—*If the diagonals of a quadrilateral are unequal and bisect each other at right angles, the figure is a rhombus.*

PROPOSITION XXXIV. THEOREM.

90. *Two parallelograms are equal when two adjacent sides and the included angle of one are respectively equal to two adjacent sides and the included angle of the other.*



Given—The parallelograms $ABCD$ and $EFGH$, with

$$AB = EF,\ AD = EH, \text{ and } \angle A = \angle E.$$

To Prove— $\square ABCD = \square EFGH.$

Dem.—Place the parallelogram $ABCD$ on $EFGH$, so that $\angle A$ coincides with its equal $\angle E$, and, since $AB = EF$, and $AD = EH$, by hypothesis, the point B will fall on F, and D on H.

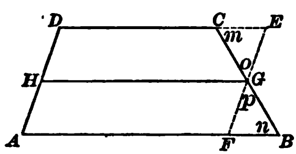
Now, since DC is parallel to AB, and HG to EF, the side DC will take the direction of HG, and C will fall somewhere on HG, or HG produced. In like manner, C will fall on FG, or FG produced.

If C falls on both HG and FG, it must fall on their intersection, G.

Hence $ABCD$ and $EFGH$ coincide throughout, and are equal. Q. E. D.

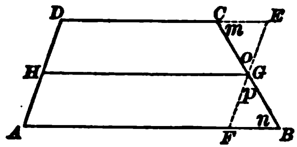
PROPOSITION XXXV. THEOREM.

91. *The straight line which joins the middle points of the non-parallel sides of a trapezoid is parallel to the parallel sides and equal to half their sum.*



Given—The line GH connecting the middle points of AD and BC of the trapezoid $ABCD$.

To Prove—HG parallel to AB and DC, and equal to $\frac{1}{2}(AB + DC)$.



Dem.—Draw EF parallel to AD through G, the middle point of BC, and produce DC to E.

In the triangles FGB and CGE, $BG = GC$, by construction,

$\angle m = \angle n$, § 37

And $\angle o = \angle p$. § 21

Hence $\triangle CGE = \triangle FGB$, § 56

And $CE = FB$, and $EG = FG$. § 45 (*a*)

In the parallelogram $AFED$, $AD = FE$. § 79

Hence DH, the half of AD, is equal and parallel to EG, the half of EF.

Therefore, $DHGE$, having two sides equal and parallel, is a parallelogram. § 83

Hence HG is parallel to DE, and also to its parallel AB.

Again, $HG = DC + CE$, § 79

And $HG = AB - FB$.

Adding these two equations, remembering that $CE = FB$, we have

$2HG = DC + AB$, Ax. 2

And $HG = \frac{1}{2}(DC + AB)$. Q. E. D.

EXERCISES.

38. The diagonals of a rectangle are equal.

39. If the diagonals of a parallelogram are equal, the figure is a rectangle.

40. The bisectors of the interior angles of a parallelogram form a rectangle.

41. If a line joining two parallels be bisected, any line drawn through the point of bisection and included between the parallels will be bisected at that point.

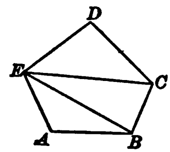
POLYGONS.

92. A **Polygon** is a plane figure bounded by straight lines.

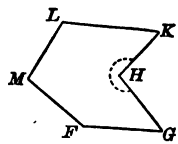
Polygons are classified according to the number of their sides.

A **Triangle,** or *Trigon*, is a polygon of three sides.

A **Quadrilateral,** or *Tetragon,* is a polygon of four sides; a *Pentagon* has five sides; a *Hexagon*, six; a *Heptagon*, seven; an *Octagon*, eight; a *Nonagon* or *Enneagon*, nine; a *Decagon*, ten; an *Undecagon*, eleven; a *Dodecagon*, twelve, etc.



93. A **Convex Polygon** is one in which no side, when produced, can enter the space enclosed by the perimeter; as *ABCDE.*



A **Concave Polygon** is one in which two or more sides, when produced, will enter the space enclosed by the perimeter; as *FGHKLM.*

The interior angle *KHG* is called a *re-entrant* angle, and the sides *KH* and *GH* are called *re-entrant* sides. A concave polygon is sometimes called a *re-entrant* polygon.

94. Parts of a Polygon.—The bounding lines *AB*, *BC*, *CD*, *DE*, *EA* are the *sides* of the polygon; and their sum is the *perimeter.*

The angles *ABC*, *BCD*, *CDE*, etc., are the *angles of the polygon.*

A *Diagonal* of a polygon is a line joining any two vertices not consecutive; as *EC*, *EB*, etc.

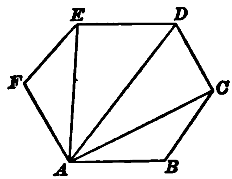
95. An *Equilateral Polygon* is a polygon whose sides are all equal. An *Equiangular Polygon* is a polygon whose angles are all equal.

96. Two polygons are *mutually equilateral* when their corresponding sides are equal. Two polygons are *mutually equiangular* when their corresponding angles are equal. In polygons that are mutually equilateral or mutually equiangular, corresponding sides or corresponding angles are *homologous*.

97. Two polygons are equal if when applied to each other they coincide throughout.

PROPOSITION XXXVI. THEOREM.

98. ***The sum of the interior angles of a polygon is equal to two right angles taken as many times, less two, as the polygon has sides.***



Given—The polygon $ABCDEF$.

To Prove—The sum of the angles equal to two right angles taken as many times as the polygon has sides, less two.

Dem.—If diagonals be drawn from the vertex A, the polygon will be divided into as many triangles as the polygon has sides, less two.

The sum of the angles of the triangles equals the sum of the angles of the polygon.

The sum of the angles of each triangle equals two right angles. § 48

Hence the sum of the angles of a polygon is equal to two right angles taken as many times as the polygon has sides, less two. Q. E. D.

If R represents a right angle, and n the number of sides, then

99. Cor. 1.—*The sum of the angles of a polygon is* $(n-2)2R$, *or* $2nR-4R$.

100. Cor. 2.—*Each angle of an equiangular polygon is* $\dfrac{(n-2)2R}{n}$.

If the right angle be regarded as unity, or 1, then

101. Cor. 3.—*The sum of the interior angles* $=(n-2)2$; *and each angle* $=\dfrac{(n-2)2}{n}$.

Scholium.—The sum of the angles of the different polygons, and the value of each angle when the polygon is equiangular, are found as follows:

Polygon.	No. of Right Angles.	Value of each Angle in Right Angles and Degrees.
Triangle	$(3-2)2=2$	$\dfrac{(3-2)2}{3}=\frac{2}{3}$, or $60°$.
Quadrilateral	$(4-2)2=4$	$\dfrac{(4-2)2}{4}=1$, or $90°$.
Pentagon	$(5-2)2=6$	$\dfrac{(5-2)2}{5}=\frac{6}{5}$, or $108°$.
Hexagon	$(6-2)2=8$	$\dfrac{(6-2)2}{6}=\frac{4}{3}$, or $120°$.

EXERCISES.

42. Required the number of degrees in each interior angle of an equiangular

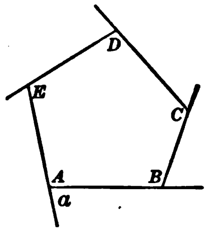
Heptagon.	Nonagon.	Undecagon.
Octagon.	Decagon.	Dodecagon.

43. Required the number of degrees in each exterior angle of an equiangular

Triangle.	Pentagon.	Heptagon.
Quadrilateral.	Hexagon.	Octagon.

Proposition XXXVII. Theorem.

102. ***If the sides of any polygon be produced so as to form one exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.***



Given—The polygon $ABCDE$, with its sides produced so as to form one exterior angle at each vertex.

To Prove—The sum of the exterior angles equal to four right angles.

Dem.—The sum of $\angle A + \angle a =$ two right angles. § 16

The same is true at each vertex.

Hence, if the polygon has n sides, the sum of the interior and exterior angles equals $2n$ right angles, or

$$\text{Int.} + \text{Ext.} = 2nR.$$

But $\qquad \text{Int.} \qquad = 2nR - 4R.$ § 99

Subtracting these two equations, we have

$$\text{Exterior angles} = 4R.$$ Q. E. D.

EXERCISES.

44. If two angles of a quadrilateral are supplementary, the other two angles are supplementary.

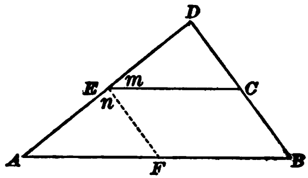
45. The exterior angles at the base of any triangle are together greater than two right angles.

46. The hypotenuse is greater than either of the other sides of a right triangle.

47. If the two exterior angles at the base of a triangle are equal, the triangle is isosceles.

PROPOSITION XXXVIII. THEOREM.

103. ***A line parallel to the base of a triangle, bisecting one side, bisects the other side also, and is equal to one half of the base.***



Given—The triangle ABD, with EC drawn from E, the middle point of AD, parallel to AB.

To Prove—$BC = DC$, and $EC = \frac{1}{2}AB$.

Dem.—Draw EF parallel to BD.

Then	$\angle D = \angle n$,	§ 37
	$\angle A = \angle m$,	§ 37
And	$AE = ED$.	Hyp.
Hence	$\triangle AEF = \triangle EDC$.	§ 56
Then	$EF = DC$.	§ 45 (*a*)
Since $BCEF$ is a parallelogram,		Cons.
Then	$BC = EF = DC$.	§ 79
In like manner $EC = FB = AF$, or $\frac{1}{2} AB$.		Q. E. D.

104. COR.—*A line joining the middle points of any two sides of a triangle is parallel to the third side.*

Given—The line EC connecting the middle points of AD and BD of the triangle ABD.

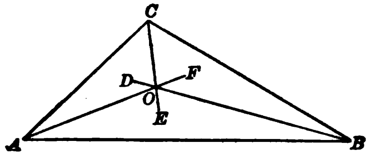
To Prove—EC parallel to AB.

Dem.—Draw EF parallel to BD.

Then EF is parallel and equal to BC.	§ 103
Hence $BCEF$ is a parallelogram.	§ 83
Therefore EC is parallel to AB.	§ 74

PROPOSITION XXXIX. THEOREM.

105. ***The bisectors of the angles of a triangle meet in a common point.***



Given—The triangle ABC, and AF, BD, and CE the bisectors of the angles A, B, and C respectively.

To Prove—That AF, BD, and CE meet in a common point.

Dem.—AF and BD will intersect at some point, as O.

Since O is in the bisector AF, it is equally distant from AB and AC. § 70

For the same reason O is equally distant from AB and BC.

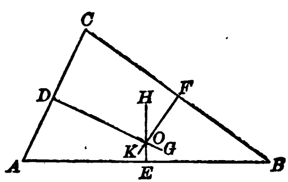
Then O is equally distant from the sides AC and BC (Ax. 1), and therefore lies in the bisector CE. § 71

Hence AF, BD, and CE meet in the common point O.

Q. E. D.

PROPOSITION XL. THEOREM.

106. ***The perpendiculars erected at the middle points of the sides of a triangle meet in a common point.***



Given—The triangle ABC, with EH, FK, and DG perpendicular at the middle points of AB, BC, and AC.

To Prove—That EH, FK, and DG meet in a common point.

Dem.—EH and FK will intersect at some point, as O.

Since O is in the perpendicular EH, it is equally distant from A and B. § 22

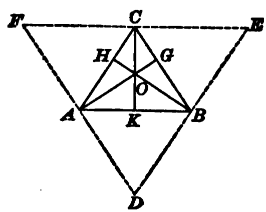
For the same reason O is equally distant from C and B.

Then O is equally distant from C and A (Ax. 1), and therefore lies in the perpendicular DG. § 24

Hence EH, FK, and DG meet in a common point. Q. E. D.

PROPOSITION XLI. THEOREM.

107. *The perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.*



Given—The triangle ABC, with CK, AG, and BH perpendicular to AB, BC, and AC respectively.

To Prove—That CK, AG, and BH meet in a common point.

Dem.—Through A, B, and C draw FD, DE, and EF parallel to BC, AC, and AB respectively.

Then $ABCF$ and $ABEC$ are parallelograms. Def.

And $FC = AB$, and $CE = AB$. § 79

Whence $FC = CE$, and C is the middle point of EF.

Now, since C is the middle point of EF, and CK is per-

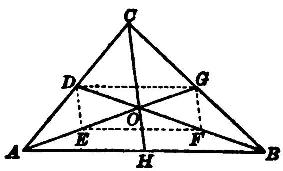
pendicular to AB, it is also perpendicular to the parallel EF at its middle point. § 36

In like manner, AG and BH are perpendiculars erected at the middle points of FD and DE respectively.

Therefore CK, AG, and BH meet in a common point, O. § 106. Q. E. D.

PROPOSITION XLII. THEOREM.

108. *The medial lines of a triangle meet in a common point.*



Given—The triangle ABC, and AG, BD, and CH the medial lines.

To Prove—That AG, BD, and CH meet in a common point.

Dem.—AG and BD will intersect at some point, as at O.

Let E and F be the middle points of AO and BO respectively, and draw DG, DE, EF, and FG.

Now, since DG bisects AC and BC, it is parallel to AB and equal to $\frac{1}{2}AB$. §§ 104, 103

Since EF bisects AO and BO, it is parallel to AB and equal to $\frac{1}{2}AB$. §§ 104, 103

Therefore DG and EF are both equal (Ax. 1) and parallel (§ 33); hence $DEFG$ is a parallelogram, § 83

And DF and EG bisect each other. § 84

By construction, E is the middle point of AO.

Therefore $AE = EO = OG$, and OG is $\frac{1}{3}AG$.

Hence BD cuts off $\frac{1}{3}$ of AG.

In like manner it may be proved that CH cuts off $\frac{1}{3}$ of AG.

Hence AG, BD, and CH meet in the common point O.

Q. E. D.

PLANE LOCI.

109. The **Locus** of a point in a plane is the *straight or curved line* which contains all points having a common property, and no others.

Thus, all points within an angle, and equally distant from its sides, lie in the bisector of the angle. § 71

Hence *the locus of a point within an angle, and equally distant from its sides, is the bisector of the angle.*

EXERCISES.

PLANE LOCI.

48. What is the locus of a point equidistant from two parallel lines?

49. What is the locus of a point equidistant from two fixed points?

50. Find the locus of a point equidistant from a fixed point.

51. What is the locus of a point equidistant from a given straight line?

52. Find the locus of a point equidistant from the extremities of a given line.

53. What is the locus of a point equidistant from the sides of an isosceles triangle?

54. What is the locus of a point equidistant from a pair of intersecting straight lines?

PRACTICAL EXAMPLES.

55. If one of the two equal angles of a triangle is 40°, what is each of the other angles?

56. If one angle of a triangle is 80°, what is each of the other angles, provided they are equal to each other?

57. If one angle of a parallelogram is 70°, what is each of the other angles?

58. What is the sum of the interior angles of a pentedecagon?

59. How many sides has the polygon in which the sum of the interior angles is five times the sum of the exterior?

EXPLANATION.—The sum of the interior angles of a polygon equals $(n-2)2$ right angles (§ 101), and the sum of the exterior angles equals four right angles (§ 102).

Hence $\qquad (n-2)2 = 5 \times 4,$

Or $\qquad 2n - 4 = 20,$

And $\qquad 2n = 24.$

$\qquad n = 12.$

Hence the polygon has twelve sides.

60. How many sides has a polygon in which the sum of the interior angles equals the sum of the exterior angles?

61. How many sides has a polygon in which the sum of the exterior angles is double the sum of the interior angles?

62. How many sides has a polygon in which the sum of the interior angles is three times the sum of the exterior angles?

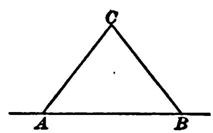
63. How many sides has a polygon in which the sum of the interior angles is ten times the sum of the exterior angles?

64. How many sides has an equiangular polygon five of whose angles are equal to eight right angles?

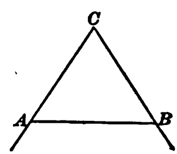
65. How many sides has an equiangular polygon seven of whose angles are equal to twelve right angles?

ORIGINAL THEOREMS.

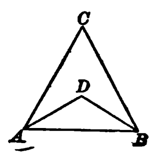
66. If the base of an isosceles triangle is produced in both directions, the obtuse angles formed will be equal.



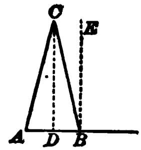
67. If the two equal sides of an isosceles triangle are produced, the obtuse angles below the base will be equal.



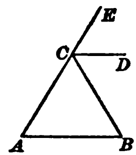
68. If the angles at the base of an isosceles triangle are bisected by lines meeting within the triangle, the triangle thus formed will be isosceles.



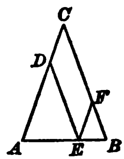
69. If the base of an isosceles triangle is produced, the exterior angle exceeds a right angle by half the vertical angle.



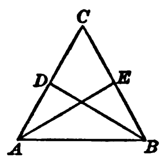
70. If an exterior angle is formed at the vertex of an isosceles triangle, its bisector is parallel to the base.



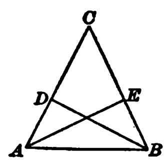
71. If from any point in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the two equal sides of the triangle.



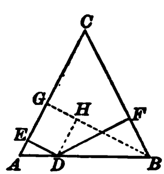
72. The lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.



73. The perpendiculars drawn from the vertices of the equal angles of an isosceles triangle are equal.



74. If from any point in the base of an isosceles triangle perpendiculars are drawn to the sides, their sum is equal to the perpendicular from the vertex to the opposite side.



75. The lines which bisect the angles at the base of an isosceles triangle and terminate in the sides are equal.

76. The perpendiculars from the middle point of the base of an isosceles triangle to the sides are equal.

77. If the non-parallel sides of a trapezoid are equal, its diagonals are equal.

78. A perpendicular let fall from one end of the base of an isosceles triangle upon the opposite side makes an angle, with the base, equal to one half of the vertical angle.

79. The middle point of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

80. If the hypotenuse of a right triangle is double one of its sides, one of the acute angles is double the other.

81. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

82. The bisectors of the angles of a rectangle form a square.

83. The medial line to any side of a triangle is less than half the sum of the other two sides.

84. The sum of the lines drawn from any point within a triangle to the vertices is less than the sum, and greater than the half sum, of the three sides.

85. The sum of the four sides of any quadrilateral is greater than the sum of the two diagonals.

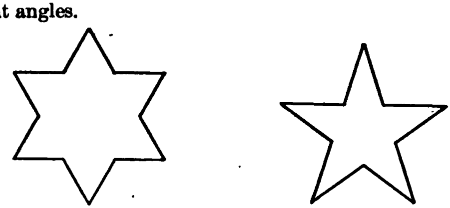
86. Prove that the number of diagonals of a polygon of n sides is

$$\frac{n(n-3)}{2}.$$

87. The altitude of a triangle divides the vertical angle into two parts whose difference is equal to the difference of the base angles of the triangle.

88. If a perpendicular be let fall from the middle point of either side of an equilateral triangle to the base, it will cut off one fourth of the base.

89. The sum of the angles at the vertices of a six-pointed star is equal to four right angles.



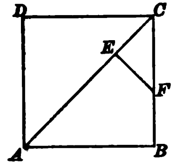
90. The sum of the angles at the vertices of a five-pointed star is equal to two right angles.

91. The sum of the angles at the vertices of an n-pointed star is equal to $2(n-4)$ right angles.

92. The bisectors of two external angles of a triangle and the bisector of the opposite internal angle meet in a point.

93. The three straight lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

94. If from the diagonal AC of a square $ABCD$, AE is cut off equal to AB, and EF is drawn perpendicular to AC, then $CE = EF = FB$.



95. If from any point in the bisector of an angle a parallel to one of the sides be drawn, the bisector, the parallel, and the remaining side form an isosceles triangle.

96. If two straight lines bisect each other, the lines joining opposite extremities are parallel.

97. If through the four vertices of a quadrilateral lines are drawn parallel to the diagonals, they will form a parallelogram twice as large as the quadrilateral.

98. If E and F are the middle points of the opposite sides, DC and AB, of a parallelogram $ABCD$, the straight lines AE and CF trisect the diagonal BD.

99. The perpendicular erected at the middle point of the base of an isosceles triangle passes through the vertex.

100. In any trapezium the line joining the middle points of the diagonals, and the line joining the middle points of the opposite sides, bisect each other.

101. The angle between the two bisectors of two consecutive angles of any quadrilateral is equal to one half the sum of the other two angles.

102. If from any point within an equilateral triangle perpendiculars to the three sides are drawn, the sum of these lines is equal to the altitude of the triangle.

103. If a quadrilateral is divided into four triangles by its diagonals, prove that the four points which are equally distant from the vertices of the respective triangles are the vertices of a parallelogram.

104. If any two points in the diagonal of a parallelogram and equidistant from its extremities be connected with the opposite vertices of the parallelogram, the figure thus formed will be a parallelogram.

BOOK II.

RATIO AND PROPORTION.

DEFINITIONS.

110. The **Ratio** of two similar quantities is the quotient obtained by dividing the first quantity by the second.

Thus, the ratio of a to b is $\frac{a}{b}$, and is expressed $a:b$.

The **Antecedent** is the first term of the ratio, and the **Consequent** is the second term.

111. A **Proportion** is an equality of ratios.

If the ratio of a to b equals the ratio of c to d, they will form a proportion, which may be written

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d}.$$

The **Couplets** of a proportion are the ratios compared.

The first and fourth terms are called the **Extremes**; the second and third terms are called the **Means**.

Thus, in the proportion $a:b=c:d$,

a and b are the *first couplet*, c and d are the *second couplet*.

a and d are *the extremes*, b and c are the *means*.

Four quantities are *reciprocally proportional* when the ratio of two values, A and B, is equal to the reciprocal of the ratio of two other values, A' and B'.

Thus, $A:B=B':A'$.

112. A Mean Proportional between two quantities is a quantity which may be made the consequent of the first couplet and the antecedent of the second.

Thus, in the proportion $a : b = b : c$, b is the *mean proportional* between a and c, and c is called the **Third Proportional.**

113. A Fourth Proportional to three quantities is the fourth term of a proportion in which the other three quantities form the first three terms.

Thus, in the proportion $a : b = c : d$, d is the *fourth proportional* to a, b, and c.

114. A Continued Proportion is one in which each consequent is the same as the next antecedent.

Thus, $$a : b = b : c = c : d = d : e.$$

115. Four quantities are in proportion by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

Thus, if $a : b = c : d$, by *alternation* $a : c = b : d$.

116. Four quantities are in proportion by **Inversion** when antecedents are made consequents, and consequents are made antecedents.

Thus, if $a : b = c : d$, by *inversion* $b : a = d : c$.

117. Four quantities are in proportion by **Composition** when the sum of antecedent and consequent is compared with either antecedent or consequent.

Thus, if $a : b = c : d$, by *composition* $a + b : b = c + d : d$.

118. Four quantities are in proportion by **Division** when the difference between antecedent and consequent is compared with either antecedent or consequent.

Thus, if $a : b = c : d$, by *division* $a - b : b = c - d : d$.

PROPOSITIONS.

PROPOSITION I. THEOREM.

119. ***In any proportion the product of the means is equal to the product of the extremes.***

If $$a : b = c : d,$$

$$\frac{a}{b} = \frac{c}{d}. \qquad \S\ 111$$

Clearing of fractions, $ad = bc$. Q. E. D.

PROPOSITION II. THEOREM.

120. ***If the product of two quantities equals the product of two other quantities, two may be made the means and two the extremes.***

Let $$ad = bc.$$

Dividing by bd, $$\frac{ad}{bd} = \frac{bc}{bd}.$$

Whence $$a : b = c : d.$$ Q. E. D.

PROPOSITION III. THEOREM.

121. ***In any proportion either extreme is equal to the product of the means divided by the other extreme.***

If $$a : b = c : d,$$

$$ad = bc. \qquad \S\ 119$$

Whence $$a = \frac{bc}{d}.$$ Q. E. D.

COR.—*Either mean is equal to the product of the extremes divided by the other mean.*

Proposition IV. Theorem.

122. ***A mean proportional between two quantities is equal to the square root of their product.***

Let $a : b = b : c.$

$b^2 = ac.$ § 119

Whence $b = \sqrt{ac}.$ Q. E. D.

Proposition V. Theorem.

123. ***If four quantities are in proportion, they will be in proportion by* Alternation.**

Let $a : b = c : d.$

$ad = bc.$ § 119

$a : c = b : d.$ § 120. Q. E. D.

Proposition VI. Theorem.

124. ***If four quantities are in proportion, they will be in proportion by* Inversion.**

Let $a : b = c : d.$

$ad = bc.$ § 119

$b : a = d : c.$ § 120. Q. E. D.

Proposition VII. Theorem.

125. ***If four quantities are in proportion, they will be in proportion by* Composition.**

Let $a : b = c : d.$

$ad = bc.$ § 119

Adding db to both members,

$ad + db = bc + db,$ Ax. 2

Or $d(a + b) = b(c + d).$

$a + b : b = c + d : d.$ § 120. Q. E. D.

PROPOSITION VIII. THEOREM.

126. *If four quantities are in proportion, they will be in proportion by* **Division.**

Let $a : b = c : d.$

$ad = bc.$ § 119

Subtracting bd from both members,

$ad - bd = bc - bd,$ Ax. 2

Or $d(a - b) = b(c - d).$

$a - b : b = c - d : d.$ § 120. Q. E. D.

PROPOSITION IX. THEOREM.

127. *If four quantities are in proportion, they will be in proportion by* **Composition and Division.**

Let $a : b = c : d.$

$$\frac{a+b}{b} = \frac{c+d}{d}. \quad (1) \quad \S\ 125$$

$$\frac{a-b}{b} = \frac{c-d}{d}. \quad (2) \quad \S\ 126$$

Dividing (1) by (2), $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}.$

Whence $a + b : a - b = c + d : c - d.$ Q. E. D.

PROPOSITION X. THEOREM.

128. *If four quantities are in proportion, like powers or like roots of those quantities will be proportional.*

Let $a : b = c : d.$

Then $\dfrac{a}{b} = \dfrac{c}{d}.$ § 111

Raising both members to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}. \quad \text{Ax. 2}$$

Whence $a^n : b^n = c^n : d^n$.

In like manner we have

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}. \qquad \text{Q. E. D.}$$

PROPOSITION XI. THEOREM.

129. *If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.*

Let $a : b = c : d$.

Then $\dfrac{a}{b} = \dfrac{c}{d}$, § 111

And $\dfrac{am}{bm} = \dfrac{cn}{dn}$.

Whence $am : bm = cn : dn$.

In like manner we have

$$\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}. \qquad \text{Q. E. D.}$$

PROPOSITION XII. THEOREM.

130. *If four quantities are in proportion, any equimultiples of the antecedents will be in proportion to any equimultiples of the consequents.*

Let $a : b = c : d$.

Then $\dfrac{a}{b} = \dfrac{c}{d}$. § 111

Multiplying by $\dfrac{m}{n}$, $\dfrac{am}{bn} = \dfrac{cm}{dn}$. Ax. 2

Whence $am : bn = cm : dn$.

In like manner we have

$$\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}. \qquad \text{Q. E. D.}$$

PROPOSITION XIII. THEOREM.

181. ***If two proportions have a couplet in each the same, the other couplets will form a proportion.***

Let $a : b = c : d.$

And $a : b = e : f,$

Then $c : d = e : f.$ Ax. 1. Q. E. D.

COR.—*If two proportions have a couplet in each proportional, the remaining couplets will be proportional.*

PROPOSITION XIV. THEOREM.

182. ***The products of the corresponding terms of two or more proportions are proportional.***

Let $a : b = c : d,$

And $m : n = p : q.$

Then $\frac{a}{b} = \frac{c}{d},$ § 111

And $\frac{m}{n} = \frac{p}{q}.$ § 111

Multiplying, $\frac{am}{bn} = \frac{cp}{dq}.$ Ax. 2

Whence $am : bn = cp : dq.$ Q. E. D.

PROPOSITION XV. THEOREM.

183. ***In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.***

Let $a : b = c : d = e : f.$

Let $r =$ the common ratio,

Then $\frac{a}{b} = r$, and $a = br.$ (1)

$\frac{c}{d} = r$, and $c = dr.$ (2)

$$\frac{e}{f} = r, \text{ and } e = fr. \qquad (3)$$

Adding equations (1), (2), and (3),

$$a + c + e = (b + d + f)r,$$

Or $$\frac{a+c+e}{b+d+f} = r = \frac{a}{b}. \qquad \text{Ax. 2}$$

Whence $$a + c + e : b + d + f = a : b. \qquad \text{Q. E. D.}$$

THE MEASUREMENT OF QUANTITIES.

134. A geometrical quantity is *measured* by finding the number of times it contains another quantity of the same kind regarded as the *unit of measure.*

Thus, if AB is the unit of measure, and CD contains AB four times, then the measure of CD is

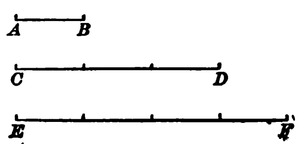
$$\frac{CD}{AB} = 4.$$

The quotient obtained by dividing a quantity by the unit of measure is called the *Numerical Measure.*

Thus, in the illustration above, 4 is the numerical measure of CD with respect to AB.

135. Two geometrical magnitudes of the same kind are *Commensurable* when there is another magnitude of the same kind which is contained an *exact* number of times in each.

Thus, if AB is the unit of measure, and is contained 3 times in CD and 4 times in EF, CD and EF are commensurable.



136. Two geometrical magnitudes of the same kind are *Incommensurable* when there is no other magnitude of the same kind which is contained an exact number of times in each.

Illustration 1.—Suppose AB and CD are incommensurable lines, then their ratio cannot be exactly expressed in numbers. If we assume a as a unit of measure which is contained in CD n times, and in AB m times, with a remainder of r, which is less than a,

Then $$\frac{AB}{CD}=\frac{ma+r}{na},$$

Or $$\frac{AB}{CD}>\frac{ma}{na}, \text{ or } \frac{m}{n}.$$

Then $\frac{m}{n}$ is called an *approximate value.*

If now we take a' as a unit of measure, and suppose that $a = na'$, then a' is contained in CD n^2 times. Let us suppose that a' is contained in AB m' times with a remainder of r', which is less than a'.

Then $$\frac{AB}{CD}=\frac{m'a'+r'}{n^2a'},$$

Or $$\frac{AB}{CD}>\frac{m'a'}{n^2a'}, \text{ or } \frac{m'}{n^2}.$$

Then $\frac{m'}{n^2}$ is a *nearer approximate value.*

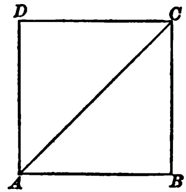
If this process be continued, we can obtain a series of variable ratios,

$$\frac{m}{n}, \frac{m'}{n^2}, \frac{m''}{n^3}, \frac{m'''}{n^4}, \text{ etc.,}$$

which will approach nearer and nearer the ratio $AB : CD$.

That is, $AB : CD$ is the *limit* of the successive approximate values.

Illustration 2.—It will be found in § 267 that if $ABCD$ is a square, then



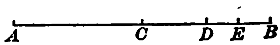
$$\frac{AC}{AB} = \sqrt{2}.$$

Since $\sqrt{2}$ can be expressed only *approximately* as a decimal, there is no line, however small, which is contained an exact number of times in both AC and AB.

Hence AC and AB are incommensurable.

THE THEORY OF LIMITS.

137. Suppose a point to move from A toward B, under the condition that the



1st second it moves over half of AB to C;
2d " " " half of the remainder to D;
3d " " " half of what now remains to E,
and so on indefinitely.

It is evident that if the point continues to move according to this law it can never reach the end of the line.

The distance from A to the moving point is an *Increasing Variable Quantity;* the distance from B to the moving point is a *Decreasing Variable Quantity;* and the distance AB is a *Constant Quantity* and the *Limit* of the increasing variable quantity.

Hence a *Variable Quantity*, or simply a *Variable*, is a quantity whose value is supposed to change. A *Constant Quantity* is a quantity which does not change its value in the same discussion. The *Limit* of a variable quantity is a constant quantity toward which the variable continues to approach, but which it can never reach.

138. As a numerical illustration, let the line *AB* (§ 137) be 2 units long. Denote the variable by x, and the difference between the variable and its limit by v; then

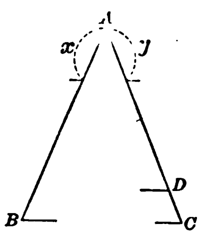
At the end of the 1st second, $x = 1$, $v = 1$;
At the end of the 2d second, $x = 1 + \frac{1}{2}$, $v = \frac{1}{2}$;
At the end of the 3d second, $x = 1 + \frac{1}{2} + \frac{1}{4}$, $v = \frac{1}{4}$,
and so on indefinitely.

It is evident that the limit of x is the limit of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc., which is 2. It is also evident that the limit of v is the last term of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, which is zero.

139. Cor.—*The difference between a variable and its limit is a new variable whose limit is zero.*

Proposition XVI. Theorem.

140. *If two variables are always equal, their limits are equal.*



Given—x and y two variables, the lim. $x = AB$, the lim. $y = AC$, and $x = y$.

To Prove— $AB = AC$.

Dem.—There are three possible suppositions:

1. $AB < AC$.
2. $AB > AC$.
3. $AB = AC$.

Suppose $AB < AC.$

On AC take $AD = AB.$

Now, since the limit of y is AC, y can assume values between AD and AC, while x cannot pass beyond AD.

That is, x and y can be unequal.

But this is contrary to the hypothesis that x and y are always equal.

Therefore AB cannot be less than AC.

In like manner we may prove that AB cannot be greater than AC.

Hence $AB = AC.$ Q. E. D.

141. Cor. 1.—*The limit of the sum of two variables is the sum of their limits.*

Let lim. $x = a$ and lim. $y = b$, then, by § 138,

$$x = a - v,$$

And $y = b - v'.$

Adding, $x + y = a + b - v - v'.$

lim. $(x + y) =$ lim. $(a + b - v - v')$. § 140

But lim. $(a + b - v - v') = a + b.$ § 139

Hence lim. $(x + y) = a + b.$

142. Cor. 2.—*The limit of the difference of two variables is the difference of their limits.*

Let lim. $x = a$ and lim. $y = b$, then, by § 138,

$$x = a - v,$$

And $y = b - v'.$

Subtracting, $x - y = a - b - v + v'.$

lim. $(x - y) =$ lim. $(a - b - v + v')$. § 140

But lim. $(a - b - v + v') = a - b.$ § 139

Hence lim. $(x - y) = a - b.$

143. Cor. 3.—*The limit of the product of two variables is the product of their limits.*

Let lim. $x = a$ and lim. $y = b$, then, by § 138,

$$x = a - v,$$

And $$y = b - v'.$$

Multiplying, $$xy = ab - bv - av' + vv'.$$

$$\text{lim. } xy = \text{lim. } (ab - bv - av' + vv'). \qquad \text{§ 140}$$

But $$\text{lim. } (ab - bv - av' + vv') = ab. \qquad \text{§ 139}$$

Hence $$\text{lim. } xy = ab.$$

144. Cor. 4.—*The limit of the quotient of two variables is the quotient of their limits.*

Let lim. $x = a$ and lim. $y = b$, then, by § 138,

$$x = a - v,$$

And $$y = b - v'.$$

Dividing, $$\frac{x}{y} = \frac{a - v}{b - v'}.$$

$$\text{lim. } \frac{x}{y} = \text{lim. } \frac{a - v}{b - v'}. \qquad \text{§ 140}$$

But $$\text{lim. } \frac{a - v}{b - v'} = \frac{a}{b}. \qquad \text{§ 139}$$

Hence $$\text{lim. } \frac{x}{y} = \frac{a}{b}.$$

145. Cor. 5.—*The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.*

Let n be any constant,

And lim. $x = a$, then, by § 138,

$$x = a - v.$$

Multiplying by n,

$$nx = an - nv.$$

	lim. $nx =$ lim. $(an - nv)$,	§ 140
But	lim. $(an - nv) = an$.	§ 139
Hence	lim. $nx = an$.	

EXERCISES.

PRACTICAL EXAMPLES.

105. Find a mean proportional between 4 and 9. Between a and b.

106. Find a third proportional to 4 and 10.

107. Find a fourth proportional to 8, 10, and 24.

108. Write $4 : x = 2 : y$ by alternation.

109. Write $8 : 6 = a : b$ by inversion.

110. Write $x - y : y = a - b : b$ by composition.

111. Write $x + y : x - y = a + b : a - b$ by division.

112. Write a proportion from $ab = c$.

113. The sum of two numbers is to their difference as 7 to 3, and their product is 160; find the numbers.

ORIGINAL THEOREMS.

114. If $a : b = c : d$, prove that $a^2 : c^2 = ab : cd$.

115. If $a : b = c : d$, prove that $a^2 : b^2 = ac : bd$.

116. If $a : b = c : d$, prove that $b - a : b = d - c : d$.

117. If $a : b = b : c$, prove that $a^2 - b^2 : b^2 - c^2 = a : c$.

118. If $a : b = b : c$, prove that $a^2 + b^2 : a^2 - b^2 = a + c : a - c$.

119. If $a : b = c : d$, prove that $ax + by : a = cx + dy : c$.

120. If $a : b = b : c = c : d = d : e$, prove that $a : e = b^4 : c^4$.

121. If $a : b = b : c = c : d$, prove that $a^3 + b^3 + c^3 : b^3 + c^3 + d^3 = a : d$.

122. Prove the lim. (.333 ...) equals $\frac{1}{3}$.

123. Prove the lim. (.555 ...) equals $\frac{5}{9}$.

124. Prove the lim. (.333 ... + .555 ...) equals $\frac{8}{9}$.

125. Prove the lim. (.555 ... − .333 ...) equals $\frac{2}{9}$.

126. If the lim. $x = a$, prove lim. $x^n = a^n$.

127. If the lim. $x = a$, prove lim. $\sqrt[n]{x} = \sqrt[n]{a}$.

128. If the lim. $x = a$, lim. $y = b$, and lim. $z = c$, prove lim. $(x + y + z) = a + b + c$.

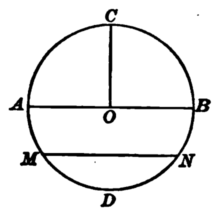
129. If the lim. $x = a$, lim. $y = b$, and lim. $z = c$, prove lim. $xyz = abc$.

BOOK III.

THE CIRCLE.

DEFINITIONS.

146. A **Circle** is a plane figure bounded by a curved line every point of which is equally distant from a point within, called the *Centre*.

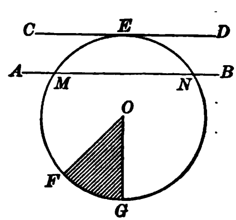


The *Circumference* is the bounding line. Any portion of the circumference is an *Arc*, as *CB*. A *Quadrant* is one fourth of a circumference.

A *Radius* is a straight line drawn from the centre to the circumference; as *OC*.

A *Diameter* is a straight line drawn through the centre, having its extremities in the circumference; as *AB*.

A *Semi-circumference* is one half of a circumference; a *Semicircle* is one half of a circle. A *Chord* is a straight line joining the extremities of an arc; as *MN*.



147. A *Tangent* is a straight line which touches the circle at only one point; as *CD*. The point *E* is called the *point of tangency*.

Two circles are *tangent to each other* when they are both tangent to the same straight line at the same point.

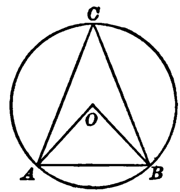
A *common tangent* to two circles is a straight line which is tangent to both of them.

A *Secant* is a straight line which cuts the circumference in two points; as *AB*.

A *Segment* of a circle is the portion included between an arc and its chord; as *MNE*.

A *Sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as *FOG*.

148. An *Inscribed Angle* is an angle whose vertex is in the circumference, and whose sides are chords; as *ACB*, *CBA*, and *BAC*.



A *Central Angle* is an angle whose vertex is at the centre, and whose sides are radii; as *AOB*.

A polygon is *inscribed in a circle* when its vertices are in the circumference and its sides are chords; as the triangle *ABC*.

A circle is *inscribed in a polygon* when its circumference touches all the sides of the polygon but does not intersect any one of them.

A polygon is *circumscribed about a circle* when its sides are tangent to the circle.

A circle is *circumscribed about a polygon* when the circumference passes through all the vertices of the polygon.

Equal circles are circles which have equal radii. For, if applied to each other, they will coincide throughout.

149. AXIOMS.

8. *All radii of a circle, or of equal circles, are equal.*

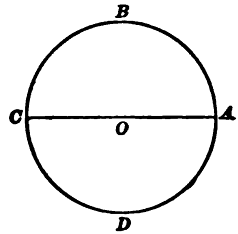
9. *The diameter of a circle is double the radius.*

10. *Every point within a circle is at a less distance from the centre than the length of the radius.*

11. *Every point without a circle is at a greater distance from the centre than the length of the radius.*

PROPOSITION I. THEOREM.

150. ***Every diameter bisects the circle and its circumference.***



Given—AC the diameter of the circle $ABCD$.

To Prove—That AC bisects the circle and its circumference.

Dem.—Apply the segment ABC to ADC by revolving it about AC as an axis. Then will the arc ABC coincide with the arc ADC, since every point of each is equally distant from the centre O.

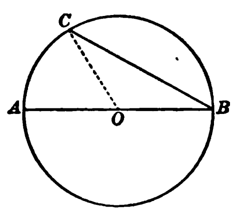
Hence the segments ABC and ADC coincide throughout, and are equal.

Therefore AC bisects the circle and its circumference.

Q. E. D.

PROPOSITION II. THEOREM.

151. ***The diameter of a circle is greater than any other chord.***



Given—AB the diameter of the circle ACB.

To Prove—AB greater than any other chord.

Dem.—Let BC be any other chord, and draw the radius CO.

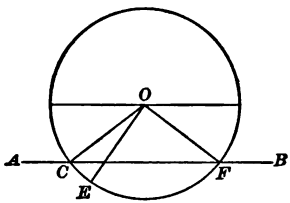
Then $OB + OC > BC$. Ax. 5

But $OB + OC = AB$. Ax. 9

Therefore $AB > BC$. Q. E. D.

PROPOSITION III. THEOREM.

152. ***A straight line cannot intersect a circumference in more than two points.***



Given—O the centre of the circle and AB a straight line intersecting the circle.

To Prove—That AB cannot intersect the circumference in more than two points.

Dem.—If AB could intersect the circumference in three points, as C, E, and F, and we should draw the radii OC, OE, and OF,

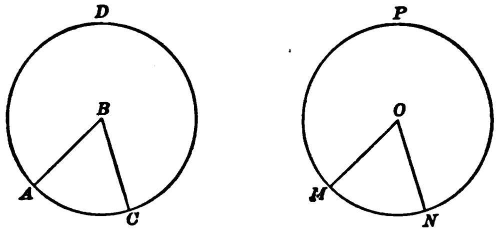
Then $OC = OE = OF$. Ax. 8

We should then have three equal straight lines drawn from the same point to a straight line, which is impossible. § 28, Cor.

Therefore AB can intersect the circumference in only two points. Q. E. D.

Proposition IV. Theorem.

158. ***In the same circle, or in equal circles, equal angles at the centre intercept equal arcs on the circumference.***



Given—B and O the centres of the circles ACD and MNP respectively, and $\angle B = \angle O$.

To Prove— Arc $AC =$ arc MN.

Dem.—Apply the sector ABC to MON, so that $\angle B$ coincides with its equal $\angle O$.

Since $AB = MO$, and $CB = NO$, Ax. 8

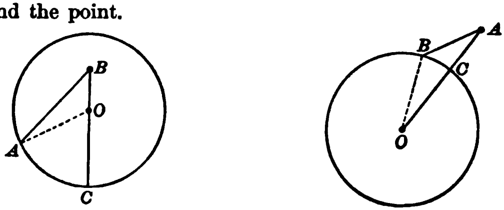
A will fall on M, and C on N,

And the arc AC will coincide with the arc MN, since all points in each are equally distant from the centre.

Therefore arc $AC =$ arc MN. § 146. Q. E. D.

EXERCISES.

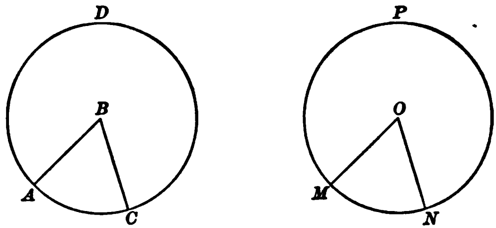
130. The longest line that can be drawn from a point within a circle to the circumference is a portion of the diameter drawn through the centre and the point.



131. The shortest line that can be drawn from a point without a circle to the circumference is the line which when produced passes through the centre of the circle.

PROPOSITION V. THEOREM.

154. CONVERSELY—***In the same circle, or in equal circles, equal arcs are intercepted by equal angles at the centre.***



Given—B and O the centres of the circles ACD and MNP respectively, and the arc $AC =$ the arc MN.

To Prove— $\angle B = \angle O$.

Dem.—Since the circles are equal, we may apply the circle ACD to MNP, so that the centre B will fall on O and the point A will fall on M; and since the arc AC is equal to the arc MN, the point C will fall on N, and BC coincides with ON. Ax. 6

Therefore the figure ABC will coincide throughout with MON,

And $\angle B = \angle O$. § 4. Q. E. D.

EXERCISES.

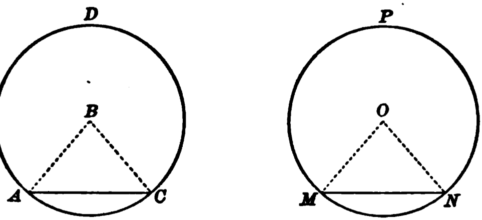
132. The shortest line that can be drawn from a point within a circle to the circumference is a portion of the diameter drawn through the point.

133. The longest line that can be drawn from a point without a circle and terminating in the concave arc is the line which passes through the centre of the circle.

134. If an equiangular triangle be constructed on each side of any triangle, the lines drawn from their outer vertices to the opposite vertices of the triangle are equal.

PROPOSITION VI. THEOREM.

155. ***In the same circle, or in equal circles, equal chords subtend equal arcs.***



Given—AC and MN equal chords in the equal circles ACD and MNP.

To Prove— arc $AC =$ arc MN.

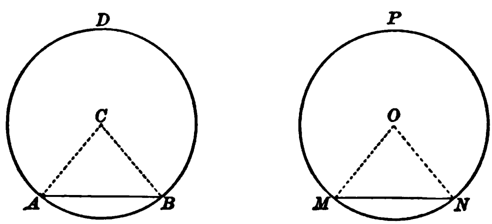
Dem.—Draw the radii AB, BC, MO, and NO.

Then in the triangles ABC and MON,

	$AC = MN$,	Hyp.
Also	$AB = MO$, and $BC = ON$.	Ax. 8
Hence	$\triangle ABC = \triangle MON$,	§ 57
And	$\angle B = \angle O$.	§ 45 (*a*)
Therefore	arc $AC =$ arc MN.	§ 153. Q. E. D.

PROPOSITION VII. THEOREM.

156. CONVERSELY—*In the same circle, or in equal circles, equal arcs are subtended by equal chords.*



Given—AB and MN equal arcs in the equal circles ABD and MNP.

To Prove— chord $AB =$ chord MN.

Dem.—Draw the radii AC, BC, MO, and NO.

Since the circles are equal and the arcs are equal,

Then $\angle C = \angle O$. § 154

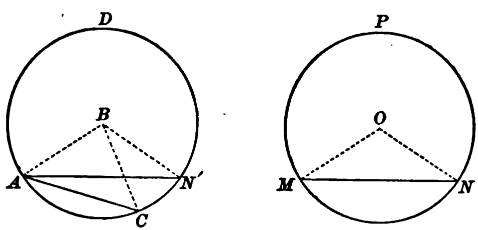
Also $AC = MO$, and $BC = NO$. Ax. 8

Hence $\triangle ABC = \triangle MON$. § 55

Therefore chord $AB =$ chord MN. § 45 (*a*). Q. E. D.

PROPOSITION VIII. THEOREM.

157. ***In the same circle, or in equal circles, the greater arc is subtended by the greater chord, each arc being less than a semi-circumference.***



Given—The equal circles ACD and MNP, and arc $MN >$ arc AC.

To Prove— chord $MN >$ chord AC.

Dem.—Draw the radii AB, BC, MO, and NO.

If we apply the circle MNP to ACD, so that the centre O falls on B, and the point M on A, then since the arc MN is greater than the arc AC, the point N will fall beyond C at N',

And $\angle ABN' > \angle ABC$. Ax. 3

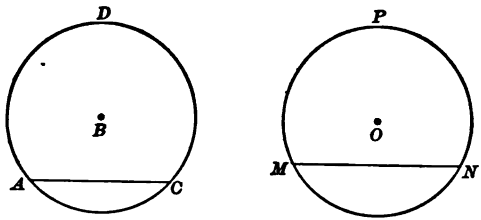
Then in the two triangles ABN' and ABC, AB is common, BC equals BN' (Ax. 8), and $\angle ABN' > \angle ABC$.

Therefore, chord AN' or its equal $MN >$ chord AC.

§ 61. Q. E. D.

PROPOSITION IX. THEOREM.

158. CONVERSELY—***In the same circle, or in equal circles, the greater chord subtends the greater arc, each arc being less than a semi-circumference.***



Given—The equal circles ACD and MNP, with the chord $MN >$ chord AC.

To Prove— arc $MN >$ arc AC.

Dem.—If the arc MN were equal to the arc AC,

chord $MN =$ chord AC. § 156

And if the arc MN were less than the arc AC,

chord $MN <$ chord AC. § 157

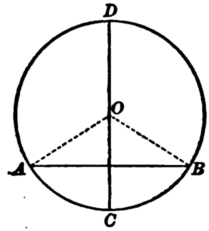
But both of these suppositions are false, since by hypothesis the chord $MN >$ chord AC.

Therefore arc $MN >$ arc AC. Q. E. D.

159. SCHOLIUM.—When each arc is greater than a semi-circumference, the greater arc is subtended by the less chord, and conversely the greater chord subtends the less arc.

PROPOSITION X. THEOREM.

160. ***The diameter perpendicular to a chord bisects the chord and the arcs subtended by it.***



Given—The circle ACB, with the diameter CD perpendicular to the chord AB.

To Prove—That CD bisects AB, and the arcs ACB and ADB.

Dem.—Draw the radii AO and BO.

Then $AO = BO$, Ax. 8

And the triangle AOB is isosceles. § 43

Then since OC is perpendicular to AB, it bisects AB and the angle AOB. § 64

Hence $\angle AOC = \angle BOC$,

And arc AC = arc BC. § 153

But arc CAD = arc CBD. § 150

Hence arc CAD − arc AC = arc CBD − arc BC,

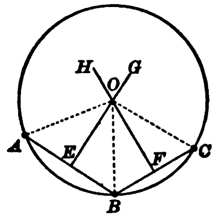
And arc AD = arc BD. Ax. 2

Therefore CD bisects chord AB and the arcs ACB and ADB. Q. E. D.

161. COR.—*The perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

PROPOSITION XI. THEOREM.

162. ***Through any three points not in the same straight line one circumference can be described, and but one.***



Given—A, B, and C, any three points not in the same straight line.

To Prove—That a circumference may be passed through them.

Dem.—Draw AB and BC, and erect EG and FH perpendicular respectively to AB and BC at their middle points.

Then, since A, B, and C are not in the same straight line, EG and FH will meet at some point, as O.

Now draw the lines AO, BO, and CO.

Then $AO = BO = CO$. § 22

Therefore a circumference described from O as a centre, with a radius equal to AO, will pass through the three points A, B, and C.

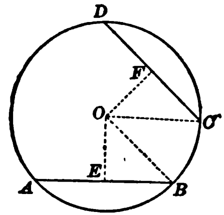
2d. Since only one perpendicular can be erected at the middle point of a line (§ 14), and two straight lines can intersect in only one point, there can be only one point equally distant from A, B, and C, and therefore only one circumference can pass through the three points. Q. E. D.

COR. 1.—*Two circumferences can intersect in only two points.*

COR. 2.—*There can be but one point equally distant from three other points.*

Proposition XII. Theorem.

163. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*



Given—AB and CD equal chords of the circle ABD.

To Prove—AB and CD equally distant from the centre.

Dem.—Draw OE and OF perpendicular to AB and CD respectively, and draw BO and CO.

Then E is the middle point of AB and F of CD, § 160

And $EB = CF$, being halves of equal chords,

And $BO = CO$. Ax. 8

Then $\triangle BOE = \triangle COF$. § 58

Whence $OE = OF$. § 45 (a)

Therefore AB and CD are equally distant from O.

Q. E. D.

EXERCISES.

135. Prove that a circumference may be circumscribed about a triangle, and also inscribed within it.

136. Prove that the inscribed and circumscribed circles of an equilateral triangle are concentric.

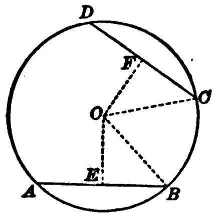
137. The radius of a circle inscribed in an equilateral triangle is one-third of the altitude of the triangle.

138. A radius which bisects a chord is perpendicular to it and bisects the arc subtended by it.

139. The radius of a circle circumscribed about an equilateral triangle is equal to twice the distance of the sides of the triangle from the centre of the circle.

PROPOSITION XIII. THEOREM.

164. CONVERSELY—*In the same circle, or in equal circles, chords equally distant from the centre are equal.*



Given—AB and CD two chords equally distant from the centre of the circle ABD.

To Prove— chord $AB =$ chord CD.

Dem.—Draw OE and OF perpendicular to AB and CD respectively, and draw BO and CO.

Then in the two right triangles BOE and COF,

$BO = CO$, Ax. 8

And, by hypothesis, $EO = FO$.

Hence $\triangle BOE = \triangle COF$, § 58

And $BE = CF$. § 45 (*a*)

But BE is one half of AB, and CF is one half of CD. § 160

Therefore $AB = CD$. Ax. 2. Q. E. D.

EXERCISES.

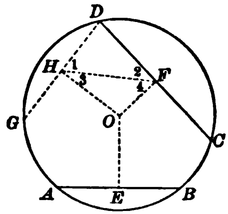
140. The perpendiculars erected at the middle points of the sides of an inscribed polygon meet in a common point.

141. A radius that bisects the arc subtended by a chord bisects the chord at right angles.

142. If an equilateral polygon be inscribed in a circle, a circumference, concentric with the given circle, may be made to pass through the middle points of all the sides.

PROPOSITION XIV. THEOREM.

165. ***In the same circle, or in equal circles, the less of two chords is at a greater distance from the centre.***



Given—The chord AB less than the chord CD in the circle ABD.

To Prove—That AB is farther from the centre than CD.

Dem.—Draw the chord DG equal to AB, and draw OE, OF, and OH perpendicular to AB, CD, and DG respectively.

Then AB, CD, and DG will be bisected at E, F, and H respectively, § 160

And $OH = OE$. § 163

Since $CD > GD$, one half of CD, or FD, $> HD$, one half of GD.

Draw HF, then, in the triangle HDF,

$$\angle 1 > \angle 2. \qquad \text{§ 68}$$

And since $\angle DHO$ and $\angle DFO$ are right angles,

$$\angle DHO - \angle 1 < \angle DFO - \angle 2,$$

Or $\angle 3 < \angle 4$,

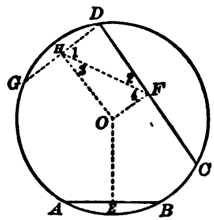
And $OF < OH$, or its equal OE. § 69

Therefore AB is farther from the centre than CD. Q. E. D.

SCHOLIUM.—In this theorem, as well as in several others of the same character, if the chords are in equal circles the demonstration is similar.

Proposition XV. Theorem.

166. Conversely—*In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.*



Given—The chord AB in the circle ABD farther from the centre than CD.

To Prove— chord $AB <$ chord CD.

Dem.—Draw the chord DG equal to AB, and draw OE, OF, and OH perpendicular to AB, CD, and DG respectively.

Then AB, CD, and DG will be bisected at E, F, and H respectively, § 160

And $OH = OE$. § 163

Draw HF, then in the triangle OHF,

$OH > OF$. Hyp.

And $\angle 3 < \angle 4$. § 68

Since $\angle DHO$ and $\angle DFO$ are right angles,

$$\angle DHO - \angle 3 > \angle DFO - \angle 4,$$

Or $\angle 1 > \angle 2$.

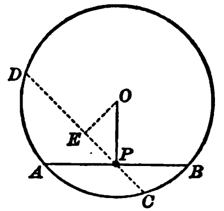
Then $DH < DF$. § 69

But DF is one half of DC, and DH is one half of DG. § 160

Hence $DG < DC$,

But $DG = AB$. Con.

Therefore chord $AB <$ chord CD. Q. E. D.



167. COR.—*The least chord that can be drawn through a given point is the chord perpendicular to the diameter through the point.*

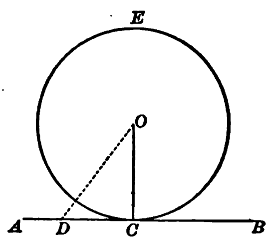
For, draw any other chord, as CD, through the given point P, and draw OE perpendicular to CD.

Then $OP > OE$. § 28

Hence AB is farther from the centre than CD, which is any other chord, and is therefore the least chord that can be drawn through the point P. § 166

PROPOSITION XVI. THEOREM.

168. *A straight line perpendicular to a radius at its extremity is tangent to the circle.*



Given—AB perpendicular to the radius CO at C.

To Prove—AB tangent to the circle.

Dem.—From the centre O draw OD to any point of AB except C.

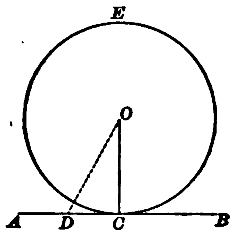
Then $OD > OC$. § 28

Hence every point of AB, except C, lies without the circle.

Therefore AB is tangent to the circle. § 147. Q. E. D.

PROPOSITION XVII. THEOREM.

169. CONVERSELY—*A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Given—AB tangent to the circle CE.

To Prove—AB perpendicular to the radius OC.

Dem.—Draw OD to any point of AB except C.

Since AB is a tangent to the circle, every point of AB, except C, lies without the circle. § 147

Hence OC is the shortest line that can be drawn from the point O to AB. Ax. 11

Therefore OC is perpendicular to AB. Ex. 23. Q. E. D.

170. COR.—*A line drawn perpendicular to a tangent at its point of contact passes through the centre of the circle.*

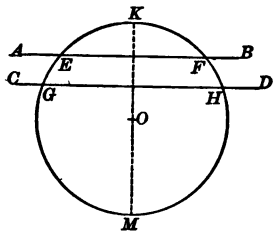
EXERCISE.

143. When two tangents meet they are equal, and the line joining the point of intersection and the centre of the circle bisects the included angle.

PROPOSITION XVIII. THEOREM.

171. *Two parallel lines intercept equal arcs on a circumference.*

CASE I.—*When both lines are secants.*



Given—AB and CD two parallel lines cutting the circle KGM whose centre is O.

To Prove— arc $EG =$ arc FH.

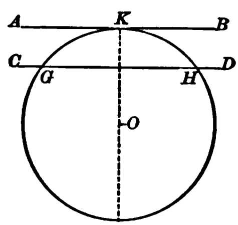
Dem.—Draw OK perpendicular to GH, then it will be perpendicular to EF also. § 36

Then arc $GK =$ arc HK, § 160

And arc $EK =$ arc FK. § 160

Subtracting, arc $EG =$ arc FH. Ax. 2. Q. E. D.

CASE II.—*When one line is a secant and the other a tangent.*



Given—AB a tangent and CD a parallel secant.

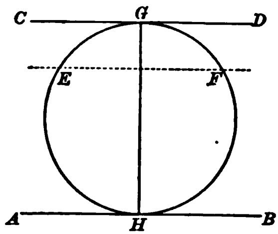
To Prove— arc $GK =$ arc HK.

Dem.—Draw OK to the point of contact; then OK is perpendicular to AB at K, § 169

And OK is perpendicular to CD also. § 36

Hence arc $GK =$ arc HK. § 160. Q. E. D.

CASE III.—*When both lines are tangents.*



Given—AB and CD two parallel tangents.

To Prove— arc HEG = arc HFG.

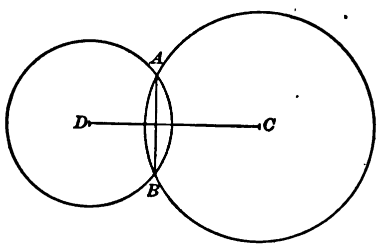
Dem.—Draw the secant EF parallel to CD.

Then	arc EG = arc FG,	Case II.
And	arc EH = arc FH.	Case II.
Adding,	arc GEH = arc GFH.	Ax. 2. Q. E. D.

172. COR.—*The straight line joining the points of contact of two parallel tangents is a diameter.*

PROPOSITION XIX. THEOREM.

173. *If two circumferences intersect each other, the line joining their centres is perpendicular to their common chord at its middle point.*



Given—D and C the centres of the two circles whose circumferences intersect at A and B.

To Prove—That DC bisects AB at right angles.

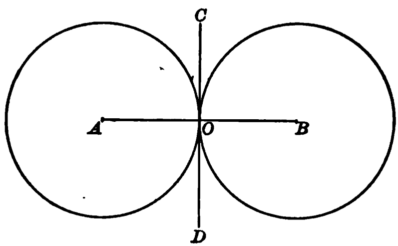
Dem.—The point D is equally distant from A and B. Ax. 8

The point C is equally distant from A and B. Ax. 8

Hence DC is perpendicular to AB at its middle point. § 25. Q. E. D.

PROPOSITION XX. THEOREM.

174. ***If two circles are tangent to each other, the line joining their centres passes through their point of contact.***



Given—A and B the centres of two circles tangent to each other at O.

To Prove—That the line AB joining their centres passes through O.

Dem.—Draw the common tangent CD.

At O erect a perpendicular to CD.

This perpendicular will pass through both centres A and B (§ 170) and coincide with AB. Ax. 6

Therefore A, O, and B are in the same straight line.

Q. E. D.

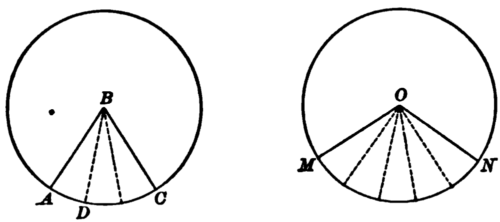
EXERCISE.

144. If two opposite sides of a quadrilateral inscribed in a circle are parallel, the other two sides are equal.

Proposition XXI. Theorem.

175. ***In the same circle, or in equal circles, two angles at the centre have the same ratio as their intercepted arcs.***

Case I.—*When the arcs are commensurable.*



Given—ABC and MON two commensurable angles, and AC and MN their intercepted arcs.

To Prove— $\dfrac{\angle ABC}{\angle MON} = \dfrac{\text{arc } AC}{\text{arc } MN}.$

Dem.—Let AD be a common unit of measure of the arcs AC and MN, and suppose it is contained 3 times in AC and 5 times in MN.

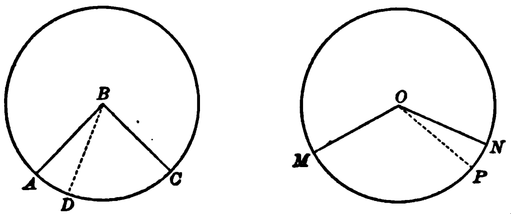
Then $\dfrac{\text{arc } AC}{\text{arc } MN} = \dfrac{3}{5}.$

If we draw radii to these points of division, the angle ABC will be divided into 3 equal parts, and MON into 5 equal parts (§ 154).

Then $\dfrac{\angle ABC}{\angle MON} = \dfrac{3}{5}.$

Hence $\dfrac{\angle ABC}{\angle MON} = \dfrac{\text{arc } AC}{\text{arc } MN}.$ Ax. 1

Case II.—*If the arcs are incommensurable.*



Given—The angles ABC and MON intercepting the incommensurable arcs AC and MN.

To Prove— $$\frac{\angle ABC}{\angle MON}=\frac{\text{arc } AC}{\text{arc } MN}.$$

Dem.—Take any arc, AD, as a unit of measure which is contained an exact number of times in AC.

Then, since AC and MN are incommensurable, AD will be contained in MN a certain number of times, with a remainder, PN, which is less than the unit of measure.

Draw OP. Since the arcs AC and MP are commensurable,

$$\frac{\angle ABC}{\angle MOP}=\frac{\text{arc } AC}{\text{arc } MP}. \qquad \text{Case I.}$$

Now, if we diminish the unit of measure indefinitely, the remainder PN will diminish indefinitely,

And MP will approach MN as its limit, and $\angle MOP$ will approach $\angle MON$ as its limit.

Hence we have the two variables

$$\frac{\angle ABC}{\angle MOP}=\frac{\text{arc } AC}{\text{arc } MP},$$

which are always equal.

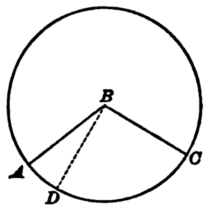
And since the limits of equal variables are equal,

Then $$\lim. \frac{\angle ABC}{\angle MOP}=\lim. \frac{\text{arc } AC}{\text{arc } MP}, \qquad \S\ 140$$

Or $$\frac{\angle ABC}{\angle MON}=\frac{\text{arc } AC}{\text{arc } MN}. \qquad \text{Q. E. D.}$$

Proposition XXII. Theorem.

176. *The numerical measure of an angle at the centre of a circle equals the numerical measure of its intercepted arc, with corresponding units of measure.*



Given—The arc AD as the unit of measure of AC, and the $\angle ABD$ the unit of measure of ABC.

To Prove—That the numerical measure of the angle ABC equals the numerical measure of the arc AC.

Dem.—We have proved that

$$\frac{\angle ABC}{\angle ABD} = \frac{\text{arc } AC}{\text{arc } AD}. \qquad \S\ 175$$

But $\dfrac{\angle ABC}{\angle ABD}$ is the numerical measure of $\angle ABC$, § 134

And $\dfrac{\text{arc } AC}{\text{arc } AD}$ is the numerical measure of arc AC. § 134

Therefore the numerical measure of $\angle ABC$ equals the numerical measure of the arc AC. Q. E. D.

177. Scholium 1.—This theorem is conventionally stated thus, "*An angle at the centre is measured by its intercepted arc.*"

In this statement the expression "*is measured by*" is used for "*has the same numerical measure as.*"

178. Scholium 2.—The *unit of measure* of arcs is $\frac{1}{360}$ part of a circumference, called a *degree*, and denoted by the symbol °. The corresponding unit of angles is $\frac{1}{360}$ of four right angles, and is also called a *degree.*

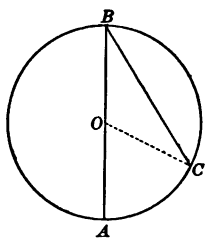
The degree (either of arc or angle) is divided into 60 equal parts called *minutes*, and the minute into 60 equal parts called *seconds*. These are denoted respectively by the symbols $'$ and $''$.

When the sum of two arcs is a quadrant, each is called the *Complement* of the other. When the sum of two arcs is a semi-circumference, each is called the *Supplement* of the other. (See § 4.)

Proposition XXIII. Theorem.

179. ***An inscribed angle is measured by one half its intercepted arc.***

Case I.—*When one side is a diameter.*



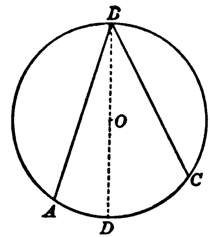
Given—AB a diameter and BC a chord of the circle ABC.

To Prove—That $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

Dem.—Draw the radius OC; then $OC = OB$,		Ax. 8
And	$\angle B = \angle C$.	§ 63
But	$\angle AOC = \angle B + \angle C$.	§ 49
Therefore	$\angle AOC = 2\angle B$,	
Or	$\angle B = \frac{1}{2}\angle AOC$.	Ax. 2
But $\angle AOC$ is measured by arc AC.		§ 177

Hence $\angle B$ is measured by $\frac{1}{2}$ arc AC.

CASE II.—*When the centre is within the angle.*



Given—BD a diameter of the circle ABC.

To Prove—That $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

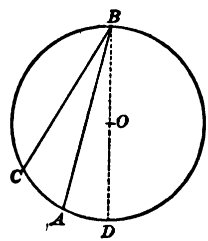
Dem.—By Case I., $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

And $\angle CBD$ is measured by $\frac{1}{2}$ arc CD.

Hence the sum of the angles ABD and CBD, or $\angle ABC$, is measured by one half the sum of the arcs AD and DC.

Therefore $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

CASE III.—*When the centre is without the angle.*



Given—BD a diameter of the circle DCB.

To Prove—That the $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

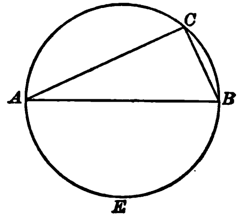
Dem.—By Case I., $\angle DBC$ is measured by $\frac{1}{2}$ arc CD,

And $\angle DBA$ is measured by $\frac{1}{2}$ arc AD.

Hence the difference of the angles DBC and DBA, or

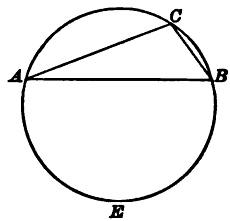
$\angle ABC$, is measured by one half the difference of the arcs DC and DA or AC.

Therefore $\angle ABC$ is measured by $\frac{1}{2}$ arc AC. Q. E. D.



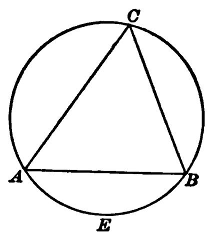
180. Cor. 1.—*Any angle inscribed in a semicircle is a right angle.*

For angle C is measured by one half of the semi-circumference AEB (§ 179), or 90°.



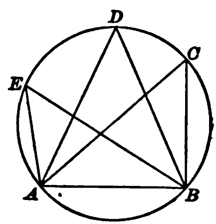
181. Cor. 2.—*An angle inscribed in a segment less than a semicircle is greater than a right angle.*

For angle C is measured by more than one half of a semi-circumference, or by an arc greater than 90°.



182. Cor. 3.—*An angle inscribed in a segment greater than a semicircle is less than a right angle.*

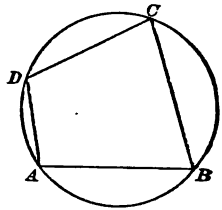
For the angle C is measured by less than one half of a semi-circumference, or by an arc less than 90°.



183. Cor. 4.—*All angles inscribed in the same segment are equal.*

For they are measured by one half of the same arc.

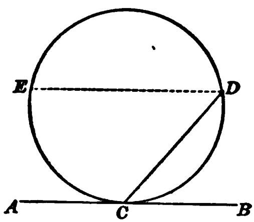
184. COR. 5.—*The opposite angles of an inscribed quadrilateral are supplements of each other.*



For the angles A and C are measured by one half of the circumference, or by an arc of 180°.

PROPOSITION XXIV. THEOREM.

185. *An angle formed by a tangent and a chord is measured by one half the arc included between its sides.*



Given—AB a tangent and CD a chord to the circle ECD.

To Prove—That $\angle DCB$ is measured by $\frac{1}{2}$ arc CD.

Dem.—Draw DE parallel to AB.

Then $\angle D = \angle DCB$, § 37

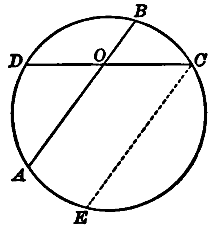
And arc EC = arc CD. § 171

But $\angle D$ is measured by $\frac{1}{2}$ arc EC. § 179

Therefore $\angle DCB$ is measured by $\frac{1}{2}$ arc EC, or its equal $\frac{1}{2}$ arc DC. Q. E. D.

PROPOSITION XXV. THEOREM.

186. *The vertical angles formed by two intersecting chords are each measured by one half of the sum of their intercepted arcs.*



Given—AB and CD two chords of the circle ADC.

To Prove—That $\angle AOD$ is measured by

$$\tfrac{1}{2}\,(\text{arc } AD + \text{arc } CB).$$

Dem.—Draw CE parallel to AB.

Then $\angle AOD = \angle C$, § 37

And arc AE = arc CB. § 171

But $\angle C$ is measured by $\tfrac{1}{2}$ arc DE. § 179

Therefore $\angle AOD$ is measured by $\tfrac{1}{2}$ arc DE, or $\tfrac{1}{2}$ (arc AD + arc AE), or $\tfrac{1}{2}$ (arc AD + arc BC). Q. E. D.

NOTE.—This theorem may be proved by drawing a line from C to A.

EXERCISES.

145. An angle at the centre of a circle contains 60°; how many degrees in the arc included between its sides?

146. An inscribed angle includes an arc of 50°; how many degrees in the angle?

147. The angles of an inscribed triangle are 40°, 60°, and 80° respectively; how many degrees in each segment of the circumference?

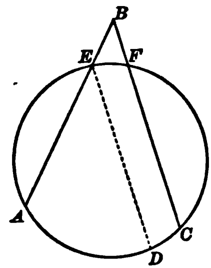
148. The triangle ABC is inscribed in a circle, $\angle A = 60°$, and the arc $AB = 100°$; find the number of degrees in each of the remaining arcs and angles.

149. The quadrilateral $ABCD$ is inscribed in a circle, $\angle A = 70°$, $\angle B = 60°$, and the arc $DC = 40°$; find the number of degrees in the remaining angles and arcs.

150. An angle formed by a chord and tangent contains 70°; find the number of degrees in each segment of the circumference.

PROPOSITION XXVI. THEOREM.

187. *The angle between two secants, intersecting without the circumference, is measured by one half the difference of the intercepted arcs.*



Given—AB and CB two secants intersecting without the circle AEC.

To Prove—That angle B is measured by

$$\tfrac{1}{2}(\text{arc } AC - \text{arc } FE).$$

Dem.—Draw ED parallel to BC.

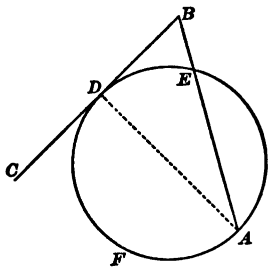
Then	$\angle B = \angle DEA$,	§ 37
And	arc DC = arc EF.	§ 171
But	$\angle DEA$ is measured by $\tfrac{1}{2}$ arc AD.	§ 179

Therefore $\angle B$ is measured by $\tfrac{1}{2}$ arc AD, or $\tfrac{1}{2}$ (arc AC — arc DC), or $\tfrac{1}{2}$ (arc AC — arc FE). Q. E. D.

NOTE.—This theorem may be proved by drawing a line from E to C; the demonstration is then similar to that of Theorem XXVII.

PROPOSITION XXVII. THEOREM.

188. *The angle between a secant and a tangent is measured by one half the difference of the intercepted arcs.*



Given—AB a secant and CB a tangent of the circle AFD.

To Prove—That $\angle B$ is measured by $\frac{1}{2}$ (arc AFD — arc DE).

Dem.—Draw AD.

Then	$\angle ADC = \angle A + \angle B$.	§ 49
Hence	$\angle B = \angle ADC - \angle A$.	Ax. 2
But	$\angle ADC$ is measured by $\frac{1}{2}$ arc AFD,	§ 185
And	$\angle A$ is measured by $\frac{1}{2}$ arc DE.	§ 179

Therefore $\angle B$ is measured by $\frac{1}{2}$ (arc AFD — arc DE).

Q. E. D.

NOTE.—This theorem may be proved by drawing a line from E parallel to BC.

EXERCISES.

151. Two chords which intersect form an angle of 45° with each other; if one of the included arcs is 30°, what is the value of the other included arc?

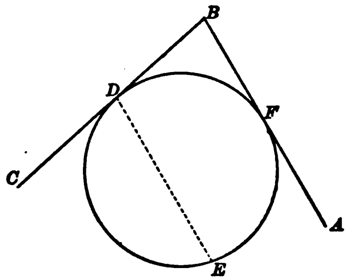
152. The arcs included between two secants are respectively $\frac{1}{18}$ and $\frac{2}{9}$ of the circumference; how many degrees in the included angle?

153. The angle included between two secants is 64 degrees, and the smaller of the intercepted arcs is $\frac{5}{18}$ of the circumference; how many degrees in the other included arc?

154. The triangle ABC is inscribed in a circle, dividing the circumference into three segments, which are to each other as 1, 3, and 5 respectively; find the value of each angle of the triangle.

PROPOSITION XXVIII. THEOREM.

189. ***The angle between two tangents is measured by one half the difference of the intercepted arcs.***



Given—*AB* and *CD* two tangents of the circle *FDE*.

To Prove—That $\angle B$ is measured by $\frac{1}{2}$ (arc *DEF* — arc *DF*).

Dem.—Draw *DE* parallel to *AB*.

Then $\angle B = \angle EDC$, § 37

And arc $DF =$ arc EF. § 171

But $\angle EDC$ is measured by $\frac{1}{2}$ arc *DE*. § 185

Therefore $\angle B$ is measured by $\frac{1}{2}$ arc *DE*, or $\frac{1}{2}$ (arc *DEF* — arc *EF*), or $\frac{1}{2}$ (arc *DEF* — arc *DF*). Q. E. D.

NOTE.—This theorem may be proved by drawing a line from *D* to *F*.

EXERCISES.

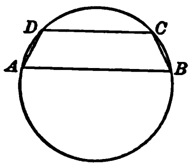
155. The quadrilateral *ABCD* is inscribed in a circle. If the sides *AB* and *AD* subtend arcs of 60° and 140° respectively, and the angle *AEB* between the diagonals is 80°, how many degrees are there in each angle of the quadrilateral?

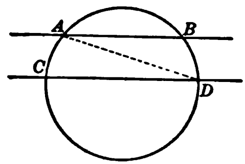
156. If the angles *A* and *B* of a circumscribed triangle *ABC* are 80° and 40°, and the sides *AB*, *BC*, and *AC* are tangent to the circle at the points *D*, *E*, and *F*, find the number of degrees in each angle of the triangle *DEF* formed by connecting *D*, *E*, and *F*.

EXERCISES.

ORIGINAL THEOREMS.

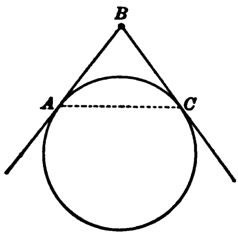
157. The lines joining the corresponding extremities of two parallel chords are equal.

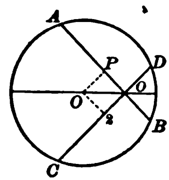




158. If two lines intercept equal arcs on the circumference of a circle, they are parallel.

159. Two tangents drawn from the same point without a circle are equal.





160. Any two chords of a circle which cut a diameter at the same point and at equal angles are equal to each other.

161. In any quadrilateral that circumscribes a circle the sums of the opposite sides are equal. (See 159.)

162. The line which bisects the angle formed by two tangents passes through the centre of the circle. (§ 70.)

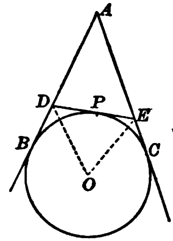
163. If a circle be inscribed in a right triangle, the difference between the two legs and the hypotenuse is equal to the diameter of the circle.

164. The circles described on any two sides of a triangle as diameters intersect on the third side.

SUGGESTION.—Let a perpendicular fall from the opposite vertex upon the third side.

165. No parallelogram but a rectangular one can be inscribed in a circle. (§ 184 and § 79.)

166. If a triangle, ADE, is formed by the intersection of three tangents to a circumference, two of which, AB and AC, are fixed, while the third, DE, is tangent to the circle at a variable point P, prove that the perimeter of the triangle ADE is equal to $AB + AC$, or $2AB$. Prove also that the angle DOE is constant.



167. If the two opposite sides of an inscribed quadrilateral are equal and parallel the figure is a rectangle.

168. The circle described on one of the equal sides of an isosceles triangle as a diameter bisects the base.

169. If two circles are tangent externally, and a straight line is drawn through their point of contact, terminating in their circumferences, the radii drawn to its extremities are parallel.

Also, the tangents at its extremities are parallel.

170. In an inscribed trapezoid the non-parallel sides are equal; and also the diagonals.

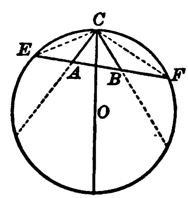
171. If a tangent is drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equally distant from the chord and the tangent.

172. If a circle is described on the radius of another circle as a diameter, any chord of the greater circle drawn through their point of contact is bisected by the circumference of the smaller circle.

173. If a circle be circumscribed about a right triangle, and another be inscribed in it, the sum of the two sides containing the right angle is equal to the sum of the diameters.

174. If a diameter and a chord be drawn from the same point in the circumference of a circle, the angle thus formed will be one-half of the angle formed by the tangents drawn through the extremities of the chord.

175. If the base of the triangle ABC be produced both ways, so that $AE = AC$ and $BF = BC$, and a circumference be described through the points E, C, and F, the line joining its centre O with the vertex C will bisect the angle C.



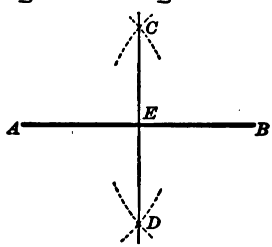
PROBLEMS OF CONSTRUCTION.

Thus far, we have assumed the construction of the figures without giving the methods, or stating the principles whereby they could be made. Indeed, the exact construction of the figures was not necessary, as they were only aids in following the demonstration of the theorems. We will now proceed to show how these and similar figures may be constructed by the application of the theorems already proved.

In the solution of the following problems we are limited to the use of the *Straight Line* and the *Circumference*, these being the only lines treated of in Elementary Geometry. The two instruments used are the *Ruler* and *Compasses*.

Proposition XXIX. Problem.

190. *To bisect a given straight line.*



Given—AB a straight line.

Required—To bisect AB.

Cons.—With A and B as centres, and with a radius greater than one half of AB, describe arcs intersecting at C and D.

Draw CD intersecting AB at E.

Then E is the middle point of AB.

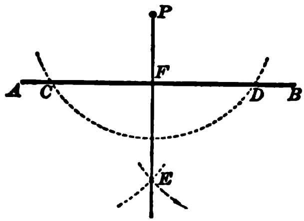
Proof.—For the line CD has two points each equally distant from A and B.

Hence CD is perpendicular to AB at its middle point.

§ 25. Q. E. F.

PROPOSITION XXX. PROBLEM.

191. ***From a given point without a straight line, to draw a perpendicular to that line.***



Given—P a point without the straight line AB.

Required—To draw a perpendicular from P to AB.

Cons.—With P as a centre, and with a radius greater than PF, describe an arc intersecting AB at C and D.

With C and D as centres, and with a radius greater than CF, describe arcs intersecting at E, and draw PE.

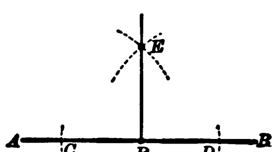
Then PE is perpendicular to AB.

Proof.—For the line PE has two points each equally distant from C and D.

Hence PE is perpendicular to CD at its middle point.
§ 25. Q. E. F.

PROPOSITION XXXI. PROBLEM.

192. ***At a given point in a straight line, to erect a perpendicular to that line.***



Given—*P* a point in the straight line *AB*.

Required—To erect a perpendicular to *AB* at *P*.

Cons.—On the line *AB* take *CP* equal to *PD*.

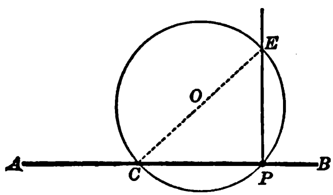
With *C* and *D* as centres, and with a radius greater than *CP*, describe arcs intersecting at *E*, and draw *EP*.

Then *PE* is perpendicular to *AB*.

Proof.—For the line *PE* has two points each equally distant from *C* and *D*.

Hence *PE* is perpendicular to *CD* at its middle point.
§ 25. Q. E. F.

Another Method.



Given—*P* a point in the straight line *AB*.

Required—To erect a perpendicular to *AB* at *P*.

Cons.—With any point *O*, without the line *AB*, as a centre, and with a radius equal to *OP*, describe a circumference intersecting *AB* at *C* and *P*.

Draw the diameter *CE*, and draw *PE*.

Then *PE* is perpendicular to *AB* at *P*.

Proof.—For ∠ *CPE* is a right angle, being inscribed in a semicircle. § 180. Q. E. F.

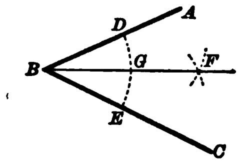
NOTE.—This method is preferred when the given point is near the end of the line.

EXERCISE.

176. Bisect a line 6 inches long.

PROPOSITION XXXII. PROBLEM.

193. ***To bisect a given angle.***



Given—ABC an angle.

Required—To bisect $\angle ABC$.

Cons.—With B as a centre, and with any convenient radius, describe an arc intersecting AB at D, and CB at E.

With D and E as centres, and with any radius greater than EG, describe arcs intersecting at F, and draw BF.

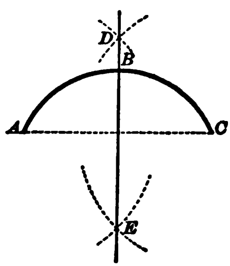
Then BF bisects $\angle ABC$.

Proof.—For, since B and F are each equally distant from E and D, BF is perpendicular to the chord of DE at its middle point. § 25

Hence BF bisects $\angle ABC$. § 160. Q. E. F.

PROPOSITION XXXIII. PROBLEM.

194. ***To bisect a given arc.***



Given—ABC an arc.

Required—To bisect the arc *ABC.*

Cons.—Draw the chord *AC.*

Bisect *AC* with the line *DE.* § 190

Then *B* is the middle point of the arc *ABC.*

Proof.—For the perpendicular erected at the middle point of a chord bisects the arc subtended by it. § 161. Q. E. F.

PROPOSITION XXXIV. PROBLEM.

195. ***At a given point in a given straight line to construct an angle equal to a given angle.***

Given—*E* a point in the given line *EF*, and *ABC* the given angle.

Required—To construct an angle at *E* equal to $\angle B$.

Cons.—With *B* as a centre, and with any convenient radius, describe an arc intersecting the sides of the angle *B* at *A* and *C.*

With *E* as a centre, and with the same radius, describe an indefinite arc *FD.*

With *F* as a centre, and with a radius equal to *AC*, describe an arc intersecting the indefinite arc *FD* at *D*, and draw *ED.*

Then $\angle E = \angle B$.

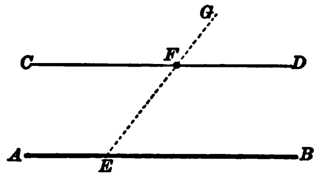
Proof.—For, by construction, the chords *AC* and *DF* are equal.

Hence arc $AC =$ arc DF. § 155

Therefore $\angle E = \angle B$. § 154. Q. E. F.

PROPOSITION XXXV. PROBLEM.

196. ***Through a given point without a straight line, to draw a parallel to the line.***



Given—F a point without the straight line AB.

Required—To draw through F a line parallel to AB.

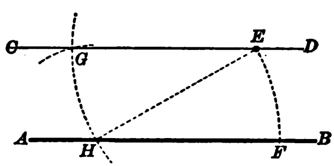
Cons.—Through F draw any line GE.

Construct $\angle GFD = \angle FEB$. § 195

Then CD is parallel to AB. § 38. Q. E. F.

Proof.—For two lines are parallel when the exterior-interior angles are equal. § 38

Another Method.



Given—E a point without the straight line AB.

Required—To draw through E a line parallel to AB.

Cons.—With the given point E as a centre, and with a radius greater than EF, describe the indefinite arc GH.

With H as a centre, and with the same radius, describe the arc EF.

With H as a centre, and with a radius equal to the chord of EF, describe an arc at G.

Draw the lines GE and HE.

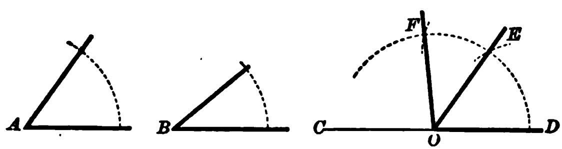
Then CD is parallel to AB.

Proof.—For $\angle GEH = \angle EHF$. § 177

And two lines are parallel when the alternate angles are equal. § 38. Q. E. F.

PROPOSITION XXXVI. PROBLEM.

197. ***Given two angles of a triangle, to find the third.***



Given—A and B two angles of a triangle.

Required—To find the third angle.

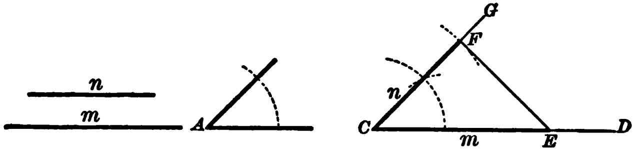
Cons.—At the point O on the indefinite straight line CD construct $\angle EOD = \angle A$, and $\angle EOF = \angle B$ (§ 195).

Then COF is the required angle.

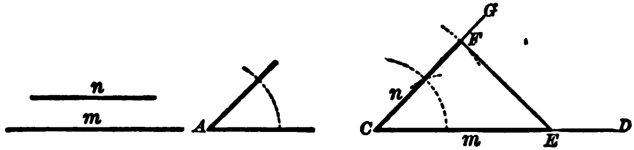
Proof.—For the third angle of a triangle equals two right angles minus the sum of the other two. § 51. Q. E. F.

PROPOSITION XXXVII. PROBLEM.

198. ***Given two sides and the included angle of a triangle, to construct the triangle.***



Given—m and n the sides and A the included angle of a triangle.



Required—To construct the triangle.

Cons.—On the line CD take CE equal to m.

Construct $\angle FCE = \angle A$. § 195

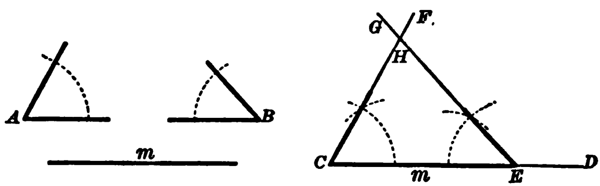
On CG take $CF = n$, and draw FE.

Then FCE is the required triangle.

Proof.—This is evident from § 55. Q. E. F.

PROPOSITION XXXVIII. PROBLEM.

199. *Given two angles and the included side of a triangle, to construct the triangle.*



Given—A and B the two angles and m the included side of a triangle.

Required—To construct the triangle.

Cons.—On the line CD take CE equal to m.

Construct $\angle C = \angle A$, and $\angle E = \angle B$. § 195

Then EG and CF intersect at H,

And CEH is the required triangle.

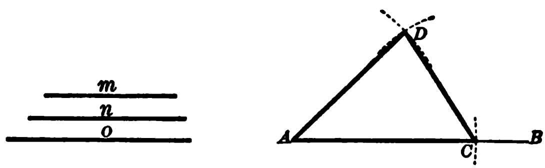
Proof.—This is evident from § 56. Q. E. F.

200. Scholium 1.—If *any* two angles of a triangle are given, the third angle may be found by § 197; hence with a side and any two angles of a triangle the construction is always possible.

201. Scholium 2.—If the sum of the two given angles is equal to or greater than two right angles, the problem is *impossible* (§ 48).

Proposition XXXIX. Problem.

202. ***Given the three sides of a triangle, to construct the triangle.***



Given—m, n, and o, the three sides of a triangle.

Required—To construct the triangle.

Cons.—On the line AB take AC equal to O.

With A as a centre, and with a radius equal to n, describe an arc at D; with C as a centre, and with a radius equal to m, describe another arc intersecting the former arc at D.

Draw AD and CD.

Then ACD is the required triangle.

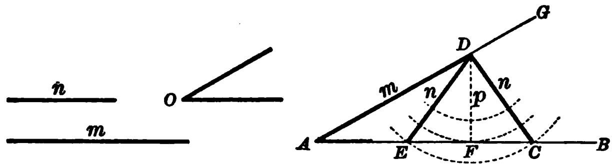
Proof.—This is evident from § 57. Q. E. F.

203. Scholium.—The problem is impossible when the sum of any two sides is equal to or less than the third side.

PROPOSITION XL. PROBLEM.

204. ***Given two sides of a triangle, and the angle opposite one of them, to construct the triangle.***

CASE I.—*When O is acute, and $n < m$.*



Given—m and n the two sides and O the angle opposite n in a triangle.

Required—To construct the triangle.

Cons.—At A, on the line AB, construct $\angle A = \angle O$, and take $AD = m$.

With D as a centre, and with a radius equal to n, describe an arc.

Then there may be three cases to this problem.

Let p be the perpendicular from D.

First.—When $n > p$.

The arc will intersect AB in two points, E and C,

And there will be two triangles, AED and ACD, either of which will answer the given conditions.

In this case there are *two solutions.*

Second.—When $n = p$.

The arc will be tangent to AB.

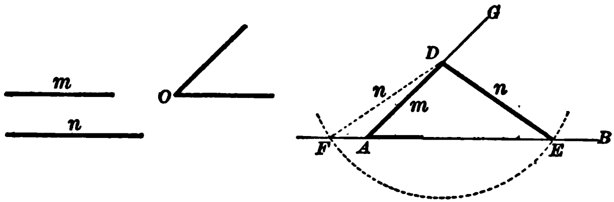
In this case there is *one solution:* a right triangle.

Third.—When $n < p$.

The arc will not intersect AB.

In this case there is *no solution.*

CASE II.—*When O is acute, and $n > m$.*



Cons.—At A, on the line AB, construct $\angle A = \angle O$, and take $AD = m$.

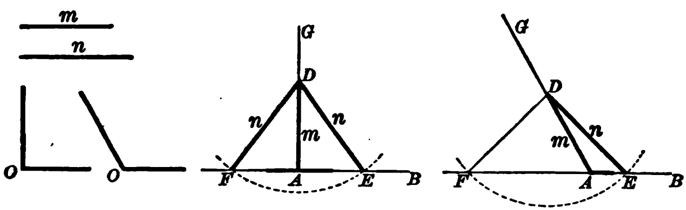
With D as a centre, and with a radius equal to n, describe an arc.

In this case the arc intersects AB in two points, E and F.

But the triangle ADF does not answer the given conditions.

There is but *one solution:* the triangle AED.

CASE III.—*When O is right or obtuse, and $n > m$.*



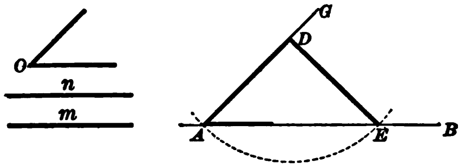
Cons.—At A, on the line AB, construct $\angle A = \angle O$, and take $AD = m$.

With D as a centre, and with a radius equal to n, describe an arc.

In this case the arc intersects AB in two points, E and F.

But in each of these cases there is evidently but *one solution.*

CASE IV.—*When O is acute, right, or obtuse, and $n = m$.*



Cons.—At A, on the line AB, construct $\angle A = \angle O$, and take $AD = m$.

With D as a centre, and with a radius equal to m or n, describe an arc.

In this case the arc intersects AB in only one point, E.

There will be but *one solution.*

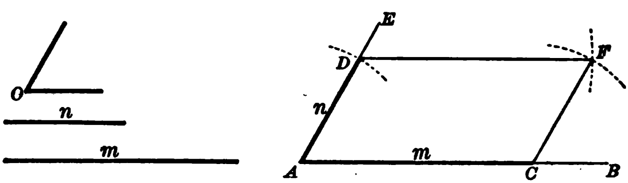
When $\angle O$ is right or obtuse, the problem is impossible.

Q. E. F.

205. SCHOLIUM.—In Case III., when the given angle is right or obtuse, and the given side opposite the angle is less than the other side, the problem is impossible.

PROPOSITION XLI. PROBLEM.

206. *Given two sides and the included angle of a parallelogram, to construct the parallelogram.*



Given—m and n the two given sides and O their included angle of a parallelogram.

Required—To construct the parallelogram.

Cons.—At A, on the line AB, construct $\angle A = \angle O$, and take $AC = m$ and $AD = n$.

With C as a centre, and with a radius equal to n, describe an arc at F; with D as a centre, and with a radius equal to m, describe an arc intersecting the former arc at F.

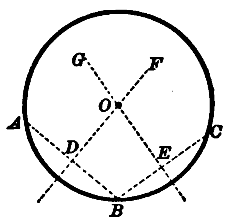
Draw CF and DF.

Then $ACFD$ is the required parallelogram.

Proof.—This is evident from § 90. Q. E. F.

PROPOSITION XLII. PROBLEM.

207. ***To find the centre of a given circumference, or of a given arc.***



Given—ABC the circumference of a circle.

Required—To find the centre of the circle.

Cons.—Take any three points of the circumference, as A, B, and C.

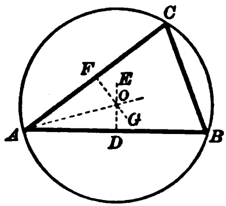
Draw AB and BC; erect DF and EG perpendicular to AB and BC, respectively, at their middle points (§ 190).

Then O, the point where these two perpendiculars intersect, is the centre required.

Proof.—This is evident from § 161. Q. E. F.

PROPOSITION XLIII. PROBLEM.

208. ***To circumscribe a circle about a given triangle.***



Given—*ABC* any triangle.

Required—To circumscribe a circle about *ABC.*

Cons.—Erect *DE* and *FG* perpendicular to *AB* and *AC*, respectively, at their middle points (§ 190).

Then *O*, the point where these two perpendiculars intersect, is the centre required.

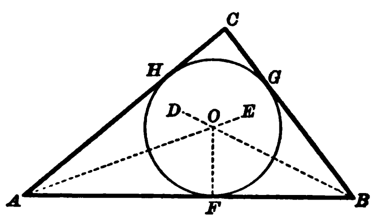
With *O* as a centre, and with a radius equal to *AO*, describe the circle *ABC.*

Then *ABC* is the required circle.

Proof.—This is evident from § 161. Q. E. F.

PROPOSITION XLIV. PROBLEM.

209. ***To inscribe a circle in a given triangle.***



Given—*ABC* any triangle.

Required—To inscribe a circle in *ABC.*

Cons.—Draw AE and BD, bisecting $\angle A$ and $\angle B$ respectively (§ 193); from O, their point of intersection, draw OF perpendicular to AB.

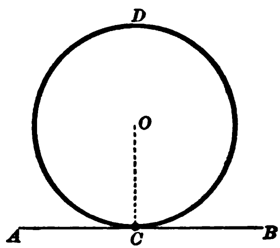
With O as a centre, and with OF as a radius, describe the circle FGH.

Then AB, BC, and AC are tangent to the circle.

Proof.—For all points on AE are equally distant from AB and AC, and all points on DB are equally distant from AB and BC; hence O is equally distant from AB, BC, and AC (§ 70). Q. E. F.

Proposition XLV. Problem.

210. ***At a given point in a given circumference, to draw a tangent to the circumference.***



Given—C a point in the circumference CD.

Required—To draw a tangent to CD at C.

Cons.—If the centre is not known, it must be found by § 207.

Draw the radius OC.

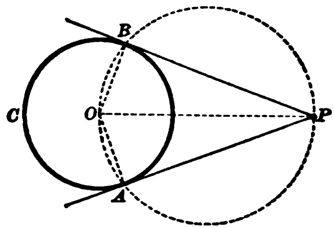
At C erect AB perpendicular to OC. § 192

Then AB is the required tangent.

Proof.—For a line perpendicular to a radius at its extremity is tangent to the circle (§ 168). Q. E. F.

Proposition XLVI. Problem.

211. ***From a given point without a given circle, to draw a tangent to the circle.***



Given—P a point without the circle ABC.

Required—To draw from P a tangent to ABC.

Cons.—Draw OP connecting the given point with the centre.

On OP as a diameter describe a circumference intersecting the given circumference at A and B.

Draw AP and BP.

Then AP and BP will both be tangent to ABC.

Proof.—For $\angle OBP$, being inscribed in a semicircle, is a right angle. § 180

Hence PB is tangent to ABC. § 168. Q. E. F.

EXERCISES.

177. To construct an angle equal to the sum of two given angles.

178. To construct an angle equal to the difference between two given angles.

179. Given an angle, to construct its complement.

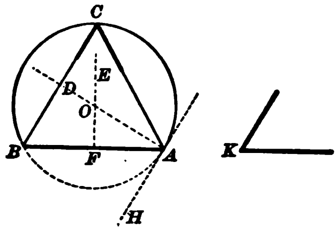
180. Given an angle, to construct its supplement.

181. To construct an angle equal to three and one half times a given angle.

182. Given a circle whose diameter is 10 inches and a point, P, 4 inches from the circumference, to draw a tangent to the circle from the point P

PROPOSITION XLVII. PROBLEM.

212. ***Upon a given straight line to describe a segment which shall contain a given angle.***



Given—AB a straight line and K any angle.

Required—To describe a segment that shall contain the angle K.

Cons.—At A construct $\angle BAH = \angle K$.

Erect AD perpendicular to AH at A, and EF perpendicular to AB at its middle point (§ 192).

With O as a centre, and with a radius equal to OA, describe the circumference ABC.

Then ACB is the required segment.

Proof.—For, since AH is a tangent, the centre of the circle will lie on AD (§ 170), and since AB is a chord, the centre lies on FE drawn perpendicular to its centre (§ 161). Hence O is the centre of the circle.

All angles inscribed in the segment ACB are equal to $\angle K$.

For $\angle C = \angle BAH$, each being measured by $\frac{1}{2}$ arc AB, § 179

And $\angle BAH = \angle K$ by construction.

Hence $\angle K = \angle C$. Ax. 1

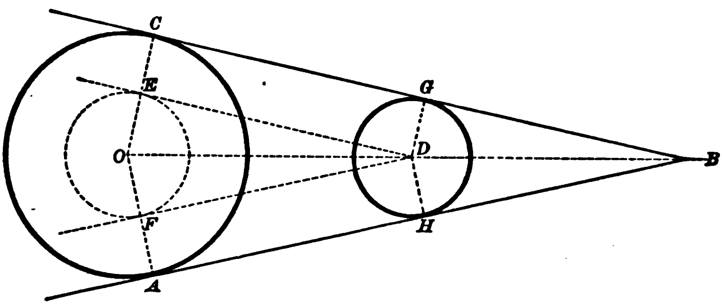
Therefore any angle inscribed in the segment ACB is equal to $\angle K$. Q. E. F.

Scholium.—If the point C falls within the segment ACB, angle C will be greater than angle K; if C falls without the segment ACB, angle C will be less than angle K; hence the arc ACB is the *locus* of the vertices of all angles equal to K whose sides pass through A and B.

Proposition XLVIII. Problem.

213. *To draw a common tangent to two given circles.*

1. To draw an exterior common tangent.



Given—AC and GH two circles.

Required—To draw an exterior common tangent.

Cons.—With O as a centre, and with a radius equal to $OC - GD$, describe the circumference EF.

From D draw a tangent to EF (§ 211).

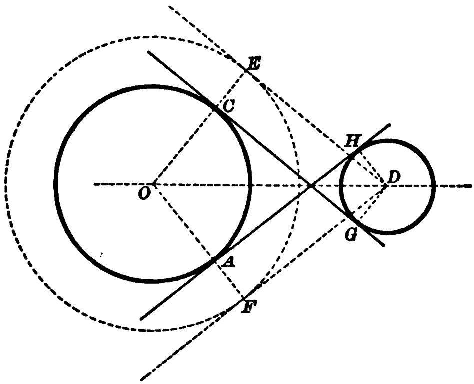
Draw OC through the point of tangency, and draw DG parallel to OC, and connect C and G.

Then CG is the required tangent.

Proof.—For, by construction, CE is equal and parallel to DG; hence $EDGC$ is a parallelogram (§ 83). But the angle E is a right angle (§ 169), and the parallelogram is a rectangle, and the angles C and G are right angles.

Therefore CG is tangent to both circles. Q. E. F.

2. To draw an interior common tangent.



Given—OC and DG two circles.

Required—To draw an interior common tangent.

Cons.—With O as a centre, and with a radius equal to $OC + DG$, describe the circumference EF.

From D draw a tangent to EF (§ 211).

Draw OE through the point of tangency, and draw DG parallel to OE, and connect C and G.

Then CG is the required tangent.

Proof.—For, by construction, DG is equal and parallel to CE; hence $EDGC$ is a parallelogram (§ 83). But the angle E is a right angle (§ 169), and the parallelogram is a rectangle, and the angles C and G are right angles.

Therefore CG is tangent to both circles. Q. E. F.

SCHOLIUM.—If two circles are tangent externally, the two interior tangents reduce to a common tangent. If they are tangent internally, the two exterior tangents reduce to a common tangent. If they intersect, only the exterior tangents are possible. If one circle is wholly within the other, the problem is impossible.

EXERCISES.

PROBLEMS FOR CONSTRUCTION.

The best method to discover the solution of a problem in Geometry is to suppose the problem solved, and construct a figure accordingly. If the problem is carefully analyzed, and auxiliary lines drawn when necessary, the solution can generally be effected.

183. Construct an angle of 45°; of 135°.

184. Construct an angle of 60°; of 30°; of 120°; of 150°.

185. Given the base and altitude of an isosceles triangle, to construct the triangle.

186. Given one of the equal sides and one of the equal angles of an isosceles triangle, to construct the triangle.

187. Given one side and the hypotenuse of a right triangle, to construct the triangle.

188. Given the hypotenuse and an acute angle of a right triangle, to construct the triangle.

189. Given the altitude and one of the equal angles of an isosceles triangle, to construct the triangle.

190. Construct a square, having given its diagonal.

191. Construct a parallelogram, having given the diagonals and their included angle.

192. Construct a circle, with a given radius, which shall pass through two given points.

193. Construct a circle, with a given radius, tangent to a given circle and to a given straight line.

194. Construct a circle, with a given radius, tangent to two given intersecting circles.

195. Inscribe in a given circle a triangle equiangular with a given triangle.

196. Draw a line tangent to a given circle and parallel to a given straight line.

197. Draw a line tangent to a circle and perpendicular to a given straight line.

198. Draw a line tangent to a given circle and making a given angle with a given straight line.

199. Trisect a right angle.

200. Construct a parallelogram, having given a side and its two diagonals.

201. In a given sector to inscribe a circle.

202. Construct a circle which shall bisect a given circumference and be tangent to a given straight line.

203. Construct an equilateral triangle, having given the radius of the inscribed circle.

204. Draw two tangents to a given circle which shall include a given angle.

205. Construct a square, having given the difference between the diagonal and the side.

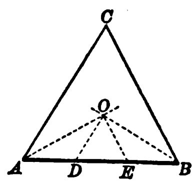
206. Through a given point A within a given angle, to draw a straight line terminated by the sides of the angle which shall be bisected at A. (See § 103.)

207. Construct a triangle, having given the base, the vertical angle, and a side.

208. Construct a triangle, having given the base, a side, and the radius of the circumscribing circle.

209. Trisect a given straight line.

Construct an equilateral triangle on AB. Bisect $\angle A$ and $\angle B$ by AO and BO respectively. Draw OD and OE respectively parallel to AC and BC. Then $AD = DE = EB$.



210. Draw a line parallel to the base of a triangle whose length is equal to the sum of the lower segments of the two sides.

LOCI OF THE CIRCLE.

211. Find the locus of all points equally distant from a fixed point.

212. Find the locus of all points equally distant from a given circumference.

213. Find the locus of the centre of a circumference which passes through two given points (§ 24).

214. Find the locus of the centre of a circumference tangent to the two sides of an angle (§ 71).

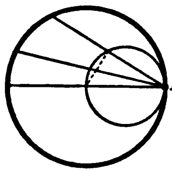
215. Find the locus of the centre of a circumference having a given radius and passing through a given point.

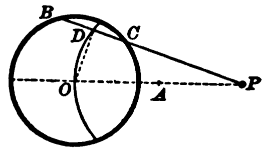
216. Find the locus of the centre of a circumference which is tangent to a given straight line at a given point.

217. Find the locus of the vertex of a right triangle having a given hypotenuse as its base (§ 180).

218. Find the locus of the middle points of all chords of equal length that can be drawn in a given circle (§ 163).

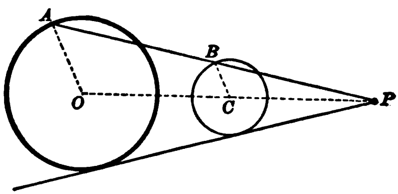
219. Find the locus of the middle points of all chords that can be drawn through a given point, *P*, in the circumference of a given circle.





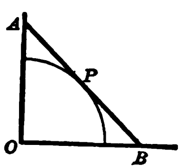
220. If from a given point, *P*, lines be drawn intersecting a given circumference in *B* and *C*, find the locus of the middle point, *D*, of the chord *BC*.

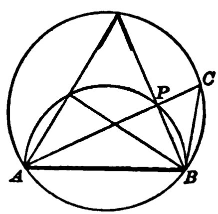
221. If from a given point, *P*, lines be drawn to any point of a given



circumference, find the locus of the middle point, *B*, of the secant *PA* (§ 103).

222. If the straight line, *AB*, of given length moves so that its ends constantly touch two fixed lines, *AO* and *BO*, which are perpendicular to each other, find the locus of the middle point *P*.





223. If upon a given base, *AB*, a triangle *ABC* is constructed, having a given vertical angle *C*, find the locus of the foot of the perpendiculars from the vertices *A* and *B* upon the opposite sides.

BOOK IV.

AREA AND RELATIONS OF POLYGONS.

DEFINITIONS.

214. The **Area** of a polygon is its ratio to another surface regarded as the unit of measure.

The *unit* for measuring polygons is generally a *square* whose side is a definite linear unit. The unit of surface is called a *superficial unit.*

215. **Equal Polygons** are polygons which when applied to each other coincide throughout.

216. **Equivalent Polygons** are polygons which have equal areas.

Equivalent polygons may differ in form, and then would not be equal. Thus a triangle, square, circle, etc., each having the area of a square foot, would be equivalent but not equal.

Equal polygons are always equivalent.

217. **Similar Polygons** are polygons which are mutually equiangular, and which have their homologous sides proportional.

Homologous sides, lines, angles, etc. of similar polygons are those which are similarly situated. Homologous lines make equal angles with the sides.

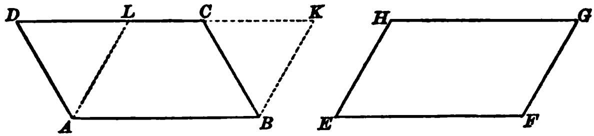
218. **Similar Arcs, Sectors,** or **Segments,** in different circles, are those whose arcs subtend equal angles at the centre.

NOTE.—The symbol ⇔ is used for the words "is equivalent to."

AREA OF POLYGONS.

Proposition I. Theorem.

219. ***Parallelograms having equal bases and equal altitudes are equivalent.***



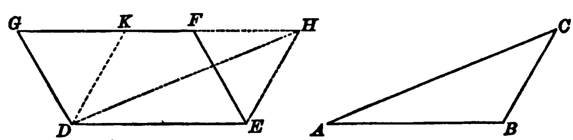
Given—*ABCD* and *EFGH* two parallelograms having equal bases and equal altitudes.

To Prove— $\square ABCD \Leftrightarrow \square EFGH$.

Dem.—Apply $\square EFGH$ to $\square ABCD$, and, since they have equal bases and equal altitudes, *EFGH* will take the position *ABKL*.

Since	$AD = BC$, and $AL = BK$,	§ 79
And	$\angle DAL = \angle CBK$,	§ 39
Then	$\triangle DAL = \triangle CBK$.	§ 55
Now, since	$ABCL + \triangle DAL = \square ABCD$,	
And	$ABCL + \triangle CBK = \square ABKL$, or $\square EFGH$,	
Therefore	$\square ABCD \Leftrightarrow \square EFGH$. Ax. 1.	Q. E. D.

220. Cor. 1.—*A triangle is equivalent to one half of a parallelogram having an equal base and an equal altitude.*



For $\triangle ABC$ is equal to one half of $\square DEHK$, having an equal base and an equal altitude. § 80

But $\square DEHK \Leftrightarrow \square DEFG$, or to any other parallelogram having an equal base and an equal altitude. § 219

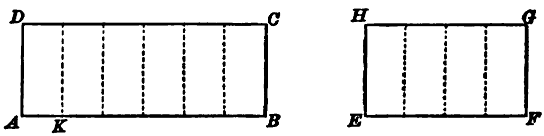
Hence $\triangle ABC$ is equivalent to one half of any parallelogram having an equal base and an equal altitude.

221. Cor. 2.—*Triangles having equal bases and equal altitudes are equivalent.*

Proposition II. Theorem.

222. ***Two rectangles having equal altitudes are to each other as their bases.***

Case I.—*When the bases are commensurable.*



Given—$ABCD$ and $EFGH$ any two rectangles having equal altitudes and commensurable bases.

To Prove— $\dfrac{ABCD}{EFGH} = \dfrac{AB}{EF}$.

Dem.—Let AK be a common unit of measure of AB and EF, and suppose it is contained 6 times in AB and 4 times in EF.

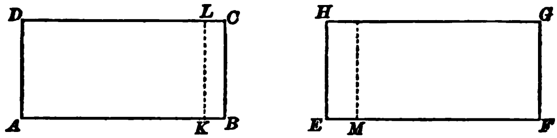
Then $\dfrac{AB}{EF} = \dfrac{6}{4}$.

If we draw lines through these points of division parallel to the sides, the rectangle $ABCD$ will be divided into 6 equal parts, and the rectangle $EFGH$ into 4 equal parts. § 219

Then $\dfrac{ABCD}{EFGH} = \dfrac{6}{4}$,

And $\dfrac{ABCD}{EFGH} = \dfrac{AB}{EF}$. Ax. 1

CASE II.—*When the bases are incommensurable.*



Given—$ABCD$ and $EFGH$ two rectangles having equal altitudes and incommensurable bases.

To Prove— $$\frac{ABCD}{EFGH} = \frac{AB}{EF}.$$

Dem.—Take any unit, as EM, which is contained an exact number of times in EF.

Then, since AB and EF are incommensurable, EM will be contained in AB a certain number of times, with a remainder, KB, which is less than the unit of measure.

Draw KL parallel to BC.

Then $$\frac{AKLD}{EFGH} = \frac{AK}{EF}.$$ Case I.

Now, if we diminish the unit of measure indefinitely, the remainder KB will diminish indefinitely,

And AK will approach AB as its limit, and $AKLD$ will approach $ABCD$ as its limit.

Hence we have two variables,

$$\frac{AKLD}{EFGH} = \frac{AK}{EF},$$

which are always equal.

And since the limits of equal variables are equal,

Then $$\lim. \frac{AKLD}{EFGH} = \lim. \frac{AK}{EF},$$ § 140

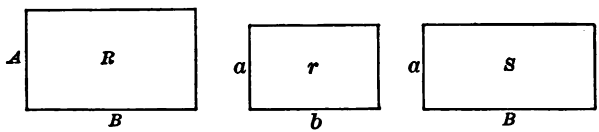
Or $$\frac{ABCD}{EFGH} = \frac{AB}{EF}.$$ Q. E. D.

223. Cor.—*Two rectangles having equal bases are to each other as their altitudes.*

Since either side of a rectangle may be taken as its base.

Proposition III. Theorem.

224. ***Any two rectangles are to each other as the products of their bases by their altitudes.***



Given—R and r any two rectangles having the bases B and b and the altitudes A and a respectively.

To Prove— $$\frac{R}{r} = \frac{B \times A}{b \times a}.$$

Dem.—Construct the rectangle S having the base B and the altitude a.

Then the rectangles R and S, having equal bases, are to each other as their altitudes.

Hence $$\frac{R}{S} = \frac{A}{a}, \qquad \text{§ 223}$$

And the rectangles r and S, having equal altitudes, are to each other as their bases.

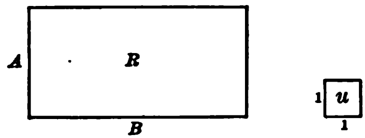
Hence $$\frac{S}{r} = \frac{B}{b}. \qquad \text{§ 222}$$

Multiplying, $$\frac{R}{r} = \frac{A \times B}{a \times b}. \qquad \text{Q. E. D.}$$

225. Scholium.—The product of two lines means the product of their *numerical measures* (§ 134).

PROPOSITION IV. THEOREM.

226. ***The area of a rectangle is equal to the product of its base and altitude.***



Given—R a rectangle whose altitude and base are A and B respectively, numerically expressed in terms of the linear unit, and let u be the unit of surface whose side is the linear unit.

To Prove—Area $R = A \times B$.

Dem.—Since any two rectangles are to each other as the products of their bases by their altitudes (§ 224),

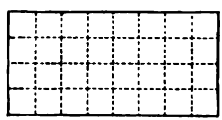
Then $$\frac{R}{u} = \frac{A \times B}{1 \times 1} = A \times B.$$

But $\frac{R}{u}$ is the area of R (§ 214).

Therefore area $R = A \times B$. Q. E. D.

227. COR.—*The area of a square is equal to the square of its side.*

228. SCHOLIUM.—When the sides of a rectangle are *commensurable*, the truth of this theorem may be seen by dividing the rectangle into squares, each equal to the unit of measure.



Thus, if the base of the rectangle contains 8 units, and the altitude 4 units, the rectangle contains 32 square units.

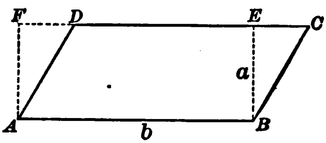
This result may be obtained in two ways: 1st. We may regard the rectangle as composed of 4 rows of squares,

each row containing 8 squares; hence the area $= 4 \times 8$ square units, or 32 square units.

2d. The number, 32, which expresses the area of the rectangle, is the product of 4 and 8, the numbers which express the lengths of the sides.

PROPOSITION V. THEOREM.

229. ***The area of a parallelogram is equal to the product of its base and altitude.***



Given—$ABCD$ a parallelogram having its base equal to b, and its altitude equal to a.

To Prove—Area $ABCD = a \times b$.

Dem.—Construct the rectangle $ABEF$.

Then	▱$ABCD \Leftrightarrow$ ▭$ABEF$.	§ 219
But	▭$ABEF = a \times b$.	§ 226
Therefore	▱$ABCD = a \times b$.	Ax. 1. Q. E. D.

230. Let P denote a parallelogram, and A and B its altitude and base respectively; and p a second parallelogram, with a and b its altitude and base respectively.

COR. 1.—*Two parallelograms are to each other as the products of their bases by their altitudes.*

For $P = A \times B$, and $p = a \times b$ (§ 229).

Dividing, $$\frac{P}{p} = \frac{A \times B}{a \times b}. \quad (1) \qquad \text{Ax. 2}$$

COR. 2.—*Two parallelograms having equal bases are to each other as their altitudes.*

For, cancelling the bases in (1), $\frac{P}{p} = \frac{A}{a}$.

COR. 3.—*Two parallelograms having equal altitudes are to each other as their bases.*

For, cancelling the altitudes in (1), $\frac{P}{p} = \frac{B}{b}$.

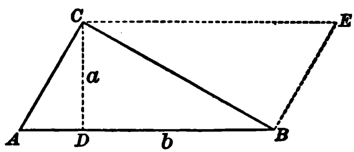
COR. 4.—*Two parallelograms having equal bases and equal altitudes are equivalent.*

For, cancelling both the bases and altitudes in (1),

$$\frac{P}{p} = 1, \text{ or } P \approx p.$$

PROPOSITION VI. THEOREM.

231. ***The area of a triangle is equal to one half the product of its base and altitude.***



Given—*ABC* a triangle having its altitude equal to *a* and its base equal to *b*.

To Prove—Area $ABC = \frac{1}{2}\, a \times b$.

Dem.—Construct the parallelogram *ABEC*.

Now,	$\triangle ABC = \frac{1}{2} \square ABEC.$	§ 220
But	$\square ABEC = a \times b.$	§ 229
Hence	$\triangle ABC = \frac{1}{2}\, a \times b.$	Q. E. D.

232. COR. 1.—*Two triangles are to each other as the products of their bases by their altitudes.*

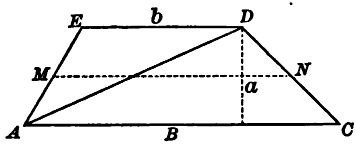
COR. 2.—*Two triangles having equal bases are to each other as their altitudes.*

COR. 3.—*Two triangles having equal altitudes are to each other as their bases.*

COR. 4.—*Two triangles having equal bases and equal altitudes are equivalent.*

PROPOSITION VII. THEOREM.

233. ***The area of a trapezoid is equal to one half the sum of its bases multiplied by its altitude.***



Given—$ACDE$ a trapezoid having its altitude equal to a, and its upper and lower bases b and B respectively.

To Prove—Area $ACDE = \frac{1}{2}(b + B)a$.

Dem.—Draw the diagonal AD, dividing the trapezoid into two triangles ACD and ADE having the common altitude a.

Then $\qquad \triangle ACD = \frac{1}{2} B \times a,$ § 231

And $\qquad \triangle ADE = \frac{1}{2} b \times a.$ § 231

Adding, $\qquad$ area $ACDE = \frac{1}{2}(B + b)a.$ Q. E. D.

234. COR.—*The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.*

Draw MN, joining the middle points of AE and CD.

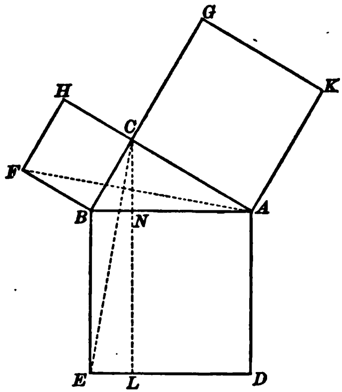
Then $\qquad MN = \frac{1}{2}(B + b).$ § 91

Hence $\qquad$ area $ACDE = MN \times a.$

SQUARES ON LINES.

PROPOSITION VIII. THEOREM.

285. ***The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.***



Given—ABC a right triangle right-angled at C.

To Prove—The square $ABED$ equivalent to the sum of the squares $BCHF$ and $ACGK$.

Dem.—Draw CL parallel to BE, and draw AF and CE.

Since $\angle BCH$ and $\angle ACB$ are right angles, ACH is a straight line (§ 20); for the same reason BCG is a straight line.

In the triangles CBE and ABF,

$AB = BE$ and $BC = BF$, sides of the same square,

And $\angle ABF = \angle CBE$,

since each is equal to a right angle + $\angle ABC$.

Hence $\triangle CBE = \triangle ABF$. § 55

Since the triangle CBE has the same base and altitude as the rectangle $BELN$,

Then $\triangle CBE \Leftrightarrow \frac{1}{2} \square BELN$. § 220

Since the triangle ABF has the same base and altitude as the square $BFHC$,

Then $\triangle ABF \Leftrightarrow \frac{1}{2}\square BFHC.$ § 220

But $\triangle CBE = \triangle ABF.$

Hence $\square BELN \Leftrightarrow \square BFHC.$

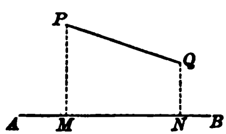
In like manner, $\square ADLN \Leftrightarrow \square ACGK.$

Adding, $\square ABED \Leftrightarrow \square BFHC + \square ACGK.$ Q. E. D.

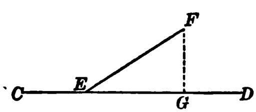
236. Cor.—*The square on either side about the right angle is equivalent to the square on the hypotenuse diminished by the square on the other side.*

Note.—This proposition has been celebrated for many centuries. It is supposed to have been first demonstrated by Pythagoras, and is therefore called the *Pythagorean Theorem.* It is the 47th proposition in the first book of Euclid's Elements, and for this reason is called the *47th of Euclid.*

237. Definitions.—The *projection of a point P* upon the line AB is the foot of the perpendicular let fall from the point upon the line.



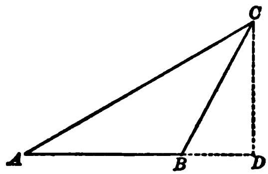
The *projection of a line PQ* upon the line AB is the distance MN included between the projections of the extremities of PQ.



If one extremity of the line EF is in the line CD, then EG is the projection of EF.

PROPOSITION IX. THEOREM.

238. *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



Given—B the obtuse angle of the triangle ABC, and draw CD perpendicular to AB produced.

To Prove—$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BD$.

Dem.—Now, $AD = AB + BD$. (1)

In the right triangle ADC,

$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2$. § 235

But $\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2 + 2AB \times BD$, (2) Sq. (1)

And $\overline{DC}^2 = \overline{BC}^2 - \overline{BD}^2$. (3) § 236

Adding (2) and (3),

$\overline{AD}^2 + \overline{DC}^2$, or $\overline{AC}^2$, $= \overline{AB}^2 + \overline{BC}^2 + 2AB \times BD$. Q. E. D.

COR.—*If the angle ABC becomes a right angle, BD becomes zero, and we have* $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

EXERCISES.

224. Required the area of a rectangle whose sides are 16 rods and 8 chains respectively.

225. Required the area of a parallelogram whose base is 6 chains and altitude 18 rods.

226. Required the area of a triangle whose base is 20 chains and altitude 660 feet.

PROPOSITION X. THEOREM.

239. ***In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.***

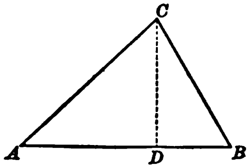


FIG. 1.

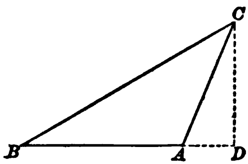


FIG. 2.

Given—B an acute angle of the triangle ABC, and draw CD perpendicular to AB.

To Prove—$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times DB.$

Dem.—In Fig. 1, $\left.\begin{array}{l} AD = AB - DB. \\ AD = DB - AB. \end{array}\right\}$ (1)
In Fig. 2,

In the right triangle ADC,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2. \qquad \S\ 235$$

But $\overline{AD}^2 = \overline{AB}^2 + \overline{DB}^2 - 2AB \times DB,$ (2) Sq. (1)

And $\overline{DC}^2 = \overline{BC}^2 - \overline{DB}^2.$ (3) § 236

Adding (2) and (3),

$\overline{AD}^2 + \overline{DC}^2$, or $\overline{AC}^2$, $= \overline{AB}^2 + \overline{BC}^2 - 2AB \times DB.$ Q. E. D.

SCHOLIUM.—The 10th Proposition can be very readily drawn from the 9th, by transposing the terms and reducing.

EXERCISES.

227. The side of an equilateral triangle is 12 inches; find the altitude and area of the triangle.

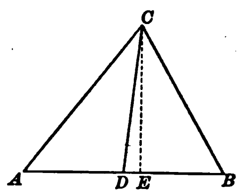
228. If the hypotenuse of a right triangle is 40 inches and the base 30 inches, find the perpendicular.

PROPOSITION XI. THEOREM.

240. *In any triangle, if a medial line be drawn from the vertex to the base,*

1. The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the medial line.

2. The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the medial line upon the base.



Given—DE the projection of the medial line upon AB.

To Prove— I. $\overline{AC}^2 + \overline{BC}^2 = 2\overline{AD}^2 + 2\overline{CD}^2$.

II. $\overline{AC}^2 - \overline{BC}^2 = 2AB \times DE$.

Dem.—In $\triangle ADC$, $\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 + 2AD \times DE$. § 238

In $\triangle DBC$, $\overline{BC}^2 = \overline{DB}^2 + \overline{DC}^2 - 2DB \times DE$. § 239

Since $AD = DB$,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 + AB \times DE. \qquad (1)$$

$$\overline{BC}^2 = \overline{AD}^2 + \overline{DC}^2 - AB \times DE. \qquad (2)$$

Adding (1) and (2), we have

$$\overline{AC}^2 + \overline{BC}^2 = 2\overline{AD}^2 + 2\overline{DC}^2.$$

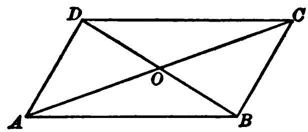
Subtracting (2) from (1), we have

$$\overline{AC}^2 - \overline{BC}^2 = 2AB \times DE.$$

Q. E. D.

Proposition XII. Theorem.

241. ***The sum of the squares on the four sides of a parallelogram is equivalent to the sum of the squares on the diagonals.***



Given—$ABCD$ a parallelogram.

To Prove—$\overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2 + \overline{DB}^2$.

Dem.—In $\triangle ADC$, $\overline{AD}^2 + \overline{DC}^2 = 2\overline{AO}^2 + 2\overline{DO}^2$. § 240

In $\triangle ABC$, $\overline{AB}^2 + \overline{BC}^2 = 2\overline{AO}^2 + 2\overline{BO}^2$. § 240

Adding, and remembering that $DO = BO$, we have

$$\overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2 = 4\overline{AO}^2 + 4\overline{BO}^2.$$

But $4\overline{AO}^2 = (2AO)^2 = \overline{AC}^2$,

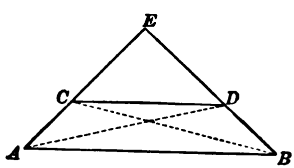
And $4\overline{BO}^2 = (2BO)^2 = \overline{BD}^2$.

Therefore $\overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2 + \overline{BD}^2$. Q. E. D.

PROPORTIONAL LINES.

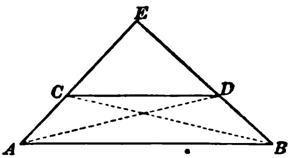
Proposition XIII. Theorem.

242. ***A parallel to one side of a triangle divides the other two sides proportionally.***



Given—CD parallel to AB.

To Prove— $\frac{CA}{CE} = \frac{DB}{DE}$.



Dem.—Draw AD and BC.

Then, since the triangles ADC and CDE have their vertices at the same point, D, and their bases in the same straight line, they have equal altitudes, and are to each other as their bases.

Whence $\frac{\triangle ADC}{\triangle CDE} = \frac{CA}{CE}$. § 232, 3.

For a similar reason the triangles BCD and DCE are to each other as their bases.

Whence $\frac{\triangle BCD}{\triangle CDE} = \frac{DB}{DE}$. § 232, 3.

But the triangles ADC and BCD have the same base, CD, and the same altitude, since CD is parallel to AB.

Whence $\triangle ADC \Leftrightarrow \triangle BCD$. § 232, 4.

Hence the two proportions have a couplet in each the same, and the remaining couplets will form a proportion.

Therefore $\frac{CA}{CE} = \frac{DB}{DE}$. § 131. Q. E. D.

243. Cor. 1.—*When a line is drawn parallel to the base of a triangle, the corresponding segments are proportional to the two sides of the triangle.*

The result of § 242 may be written,

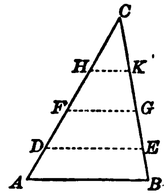
$$CA : CE = DB : DE.$$

By composition, $CA + CE : CE = DB + DE : DE$, § 125

Or $AE : CE = BE : DE$,

Also $AE : AC = BE : BD$.

244. COR. 2.—*Straight lines drawn parallel to the base of a triangle divide the other sides proportionally.*



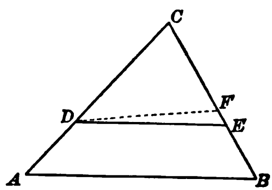
From this theorem it is evident that

$$\frac{CH}{CK} = \frac{HF}{KG} = \left(\frac{CF}{CG}\right) = \frac{FD}{GE} = \left(\frac{CD}{CE}\right) = \frac{DA}{EB}.$$

Dropping the ratios in parentheses, the corresponding segments are proportional.

PROPOSITION XIV. THEOREM.

245. CONVERSELY—***A line which divides two sides of a triangle proportionally is parallel to the third side.***



Given—DE a line drawn so that

$$\frac{AC}{DC} = \frac{BC}{EC}.$$

To Prove—DE parallel to AB.

Dem.—If DE is not parallel to AB, draw DF parallel.

Then $\frac{AC}{DC} = \frac{BC}{FC}.$ § 242

But, by hypothesis, $\frac{AC}{DC} = \frac{BC}{EC}.$

Then $\frac{BC}{FC} = \frac{BC}{EC},$ Ax. 1

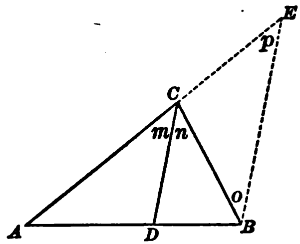
Or $FC = EC.$

Then DE must coincide with DF, and is parallel to AB.

Q. E. D.

PROPOSITION XV. THEOREM.

246. ***In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.***



Given—CD bisecting the angle ACB.

To Prove— $\frac{DA}{DB} = \frac{CA}{CB}$.

Dem.—Draw BE parallel to DC, and produce AC to E.

Then $\angle n = \angle o$, § 37, 1

And $\angle m = \angle p$. § 37, 2

But, by hypothesis, $\angle m = \angle n$.

Then $\angle o = \angle p$, Ax. 1

And $CE = CB$. § 66

Now, since CD is parallel to BE,

Then $\frac{DA}{DB} = \frac{CA}{CE}$. § 242

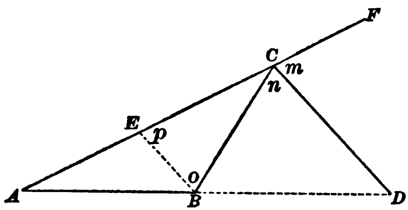
Putting CB for its equal CE, we have

$$\frac{DA}{DB} = \frac{CA}{CB}. \qquad \text{Q. E. D.}$$

247. COR.—CONVERSELY.—*If a line is drawn from any angle of a triangle dividing the opposite side proportional to the adjacent sides, it bisects the angle.*

PROPOSITION XVI. THEOREM.

248. ***In any triangle, the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.***



Given—CD bisecting the angle BCF.

To Prove— $\dfrac{DA}{DB} = \dfrac{CA}{CB}$.

Dem.—Draw BE parallel to DC.

Then $\angle n = \angle o$, § 37, 1

And $\angle m = \angle p$. § 37, 2

But, by hypothesis, $\angle m = \angle n$.

Then $\angle o = \angle p$, Ax. 1

And $CE = CB$. § 66

Now, since BE is parallel to DC,

Then $\dfrac{DA}{DB} = \dfrac{CA}{CE}$. § 242

Putting CB for its equal CE, we have

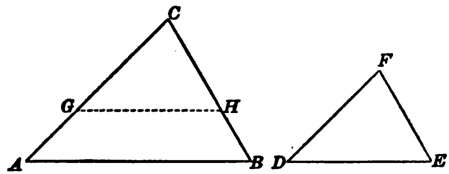
$$\frac{DA}{DB} = \frac{CA}{CB}. \qquad \text{Q. E. D.}$$

249. COR.—CONVERSELY.—*If a line is drawn from any angle of a triangle dividing the opposite side externally proportional to the adjacent sides, it bisects the exterior angle.*

SIMILAR POLYGONS.

PROPOSITION XVII. THEOREM.

250. ***Triangles which are mutually equiangular are similar.***



Given—The triangles ABC and DEF, with

$$\angle A = \angle D, \quad \angle B = \angle E, \text{ and } \angle C = \angle F.$$

To Prove—$\triangle ABC$ and $\triangle DEF$ similar.

Dem.—Place the triangle DEF upon ABC, so that $\angle F$ coincides with $\angle C$, and $\triangle DEF$ will take the position of $\triangle GHC$.

Since $\angle CGH = \angle A$, GH is parallel to AB, § 38, 2

And $$\frac{AC}{GC} = \frac{BC}{HC},$$ § 242

Or $$\frac{AC}{DF} = \frac{BC}{EF}.$$

In a similar manner it may be shown that

$$\frac{AC}{DF} = \frac{AB}{DE}.$$

Hence $\triangle ABC$ and $\triangle DEF$ have their homologous sides proportional, and are similar. § 217. Q. E. D.

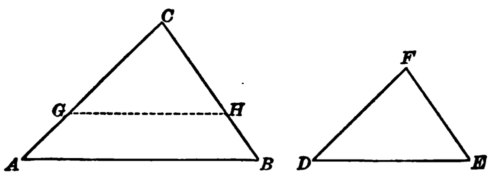
251. COR. 1.—*Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

252. COR. 2.—*Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

253. Cor. 3.—*If two triangles are similar to a third triangle, they are similar to each other.*

Proposition XVIII. Theorem.

254. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



Given—The triangles ABC and DEF, with

$$\angle C = \angle F, \text{ and } \frac{AC}{DF} = \frac{BC}{EF}.$$

To Prove—$\triangle ABC$ and $\triangle DEF$ similar.

Dem.—Place the triangle DEF upon $\triangle ABC$, so that $\angle F$ coincides with $\angle C$, and $\triangle DEF$ will take the position of $\triangle GHC$.

Then, by hypothesis, $\dfrac{AC}{GC} = \dfrac{BC}{HC}$.

Hence GH is parallel to AB, § 245

And $\angle CGH = \angle A$, and $\angle CHG = \angle B$. § 37

Then $\triangle ABC$ and $\triangle GHC$, or its equal $\triangle DEF$, are similar. § 250. Q. E. D.

EXERCISES.

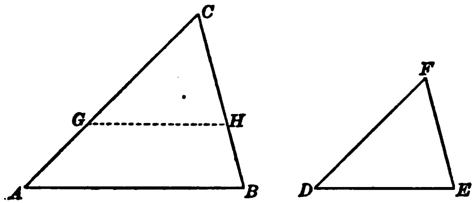
229. In the triangle ABC, $AB = 21$, $AC = 16$, and $BC = 12$; find the segments of AB formed by the bisector of the angle C.

230. In the triangle ABC, $AC = 30$, $BC = 10$, and $AB = 30$; find the segments of AB formed by the bisector of the external angle at C.

231. If the sides of a triangle are a, b, and c, find the sides of a similar triangle whose base is d.

PROPOSITION XIX. THEOREM.

255. *Two triangles are similar when their homologous sides are proportional.*



Given—The triangles ABC and DEF, with

$$\frac{AC}{DF} = \frac{AB}{DE} = \frac{BC}{EF}.$$

To Prove—$\triangle ABC$ and $\triangle DEF$ similar.

Dem.—Take $CG = DF$, and $CH = EF$, and draw GH.

Then $\triangle ABC$ and $\triangle GHC$ are similar. § 254

Hence $\frac{AC}{GC} = \frac{AB}{GH}$, or $\frac{AC}{DF} = \frac{AB}{GH}$. § 217

But, by hypothesis, $\frac{AC}{DF} = \frac{AB}{DE}$.

Hence $\frac{AB}{DE} = \frac{AB}{GH}$, or $DE = GH$, Ax. 1

And $\triangle DEF = \triangle GHC$. § 57

Therefore the triangles ABC and DEF are similar.

Q. E. D.

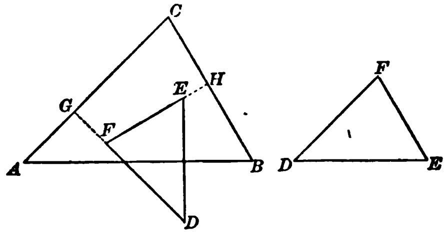
256. COR.—*The altitudes of two similar triangles are to each other as any two homologous sides.*

EXERCISE.

232. In the triangle ABC, $AC = 24$ in. and $BC = 30$ in.; if a line parallel to the base AB cuts the side AC 8 in. from the base, how far from the vertex C does it cut the side BC?

PROPOSITION XX. THEOREM.

257. ***Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.***



Given—The triangles ABC and DEF, with their sides parallel each to each, or perpendicular each to each.

To Prove—$\triangle ABC$ and $\triangle DEF$ similar.

Dem.—When the sides of two angles are parallel each to each, or perpendicular each to each, these angles are either equal or supplementary (§§ 39, 40).

Hence we may make the following suppositions with regard to the angles of these triangles:

1. $A + D = 2$ rt. $\angle s$, $B + E = 2$ rt. $\angle s$, $C + F = 2$ rt. $\angle s$.
2. $A = D$, $B + E = 2$ rt. $\angle s$, $C + F = 2$ rt. $\angle s$.
3. $A = D$, $B = E$, $C = F$. § 54

But since the sum of the angles of two triangles cannot exceed four right angles (§ 48), only the third supposition is possible.

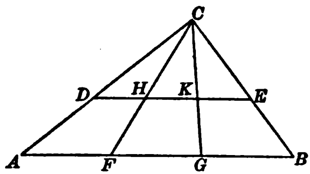
Hence the triangles ABC and DEF are mutually equiangular.

Therefore $\triangle ABC$ and $\triangle DEF$ are similar. § 250. Q. E. D.

258. SCHOLIUM.—In similar triangles whose sides are parallel each to each, or perpendicular each to each, two parallel sides, or two perpendicular sides, are homologous, and the angles included by homologous sides are equal.

PROPOSITION XXI. THEOREM.

259. ***The lines drawn from the vertex of a triangle to the base divide the base and a parallel to it proportionally.***



Given—DE parallel to AB, in the triangle ABC, and CF and CG drawn from the vertex to the base.

To Prove— $\frac{AF}{DH}=\frac{FG}{HK}=\frac{GB}{KE}$.

Dem.—Since $\triangle DHC$ is similar to $\triangle AFC$, $\triangle HKC$ to $\triangle FGC$, and $\triangle KEC$ to $\triangle GBC$, § 250

Then $\frac{AF}{DH}=\left(\frac{FC}{HC}\right)=\frac{FG}{HK}=\left(\frac{GC}{KC}\right)=\frac{GB}{KE}$. § 217

Dropping the ratios in parentheses, the base and parallel are divided proportionally. Q. E. D.

260. COR.—*If AB is divided into equal parts at F and G, then DE is divided into equal parts at H and K.*

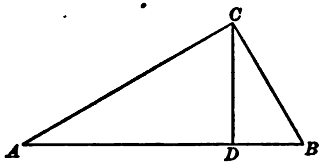
PROPOSITION XXII. THEOREM.

261. ***If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,***

I. The triangles formed are similar to the whole triangle, and to each other.

II. The perpendicular is a mean proportional between the segments of the hypotenuse.

III. Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.



Given—CD perpendicular to the hypotenuse of the right triangle ABC.

To Prove, 1.—The triangles ADC and BDC similar to the triangle ABC and to each other.

Dem.—In the right triangles ADC and ABC, $\angle A$ is common.

Hence $\triangle ADC$ and $\triangle ABC$ are similar. § 252

In like manner, $\triangle BDC$ and $\triangle ABC$ are similar.

Hence $\triangle ADC$ and $\triangle BDC$ are similar. § 253

To Prove, 2.— $AD : DC = DC : DB.$

Dem.—Since the triangles ADC and BDC are similar, their homologous sides are proportional (§ 217).

Hence $AD : DC = DC : DB.$ (1)

To Prove, 3.— $AB : AC = AC : AD.$

Dem.—Since the triangles ABC and ADC are similar, their homologous sides are proportional (§ 217).

Hence $AB : AC = AC : AD.$ (2)

In like manner, $AB : BC = BC : BD.$ (3) Q. E. D.

262. Cor. 1.—*The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

From (1), $\overline{DC}^2 = AD \times DB.$ (4) § 119

263. COR. 2.—*The square of either leg is equal to the product of the hypotenuse and the adjacent segment.*

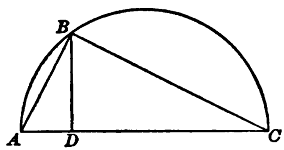
From (2), $\overline{AC}^2 = AB \times AD.$ (5) § 119

From (3), $\overline{BC}^2 = AB \times DB.$ (6) § 119

264. COR. 3.—*The squares of the legs are proportional to the adjacent segments of the hypotenuse.*

Dividing (5) by (6), $\dfrac{\overline{AC}^2}{\overline{BC}^2} = \dfrac{AB \times AD}{AB \times DB} = \dfrac{AD}{DB}.$ Ax. 2.

265. If from any point in a circumference a perpendicular be let fall, and chords be drawn to the extremities of the diameter, then ABC is a right triangle (§ 180), and

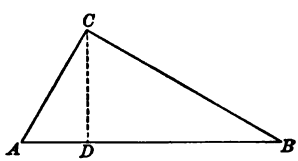


1. *The perpendicular is a mean proportional between the segments of the diameter.*

2. *Each chord is a mean proportional between the diameter and the segment adjacent to that chord.*

PROPOSITION XXIII. THEOREM.

266. *In any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.*



Given—AB the hypotenuse of the right triangle ABC.

To Prove— $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$

Dem.—Draw CD perpendicular to AB.

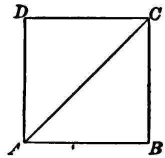
Then $\overline{AC}^2 = AD \times AB,$ § 263

And $\overline{BC}^2 = DB \times AB.$ § 263

Adding, $\overline{AC}^2 + \overline{BC}^2 = (AD + DB)AB,$

Or $\overline{AC}^2 + \overline{BC}^2 = AB \times AB = \overline{AB}^2.$ Q. E. D.

267. COR.—*Let ABCD be a square.*



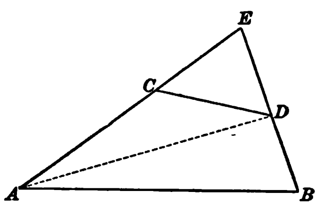
Then $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2,$

Or $\frac{\overline{AC}^2}{\overline{AB}^2} = 2$, and $\frac{AC}{AB} = \sqrt{2}.$

Hence *the diagonal of a square is incommensurable with its side.* § 136

PROPOSITION XXIV. THEOREM.

268. *Two triangles which have an angle in each equal are to each other as the product of the sides including those equal angles.*



Given—The triangles ABE and CDE having the common angle E.

To Prove— $\frac{\triangle ABE}{\triangle CDE} = \frac{AE \times BE}{CE \times DE}.$

Dem.—Draw AD.

Now, since the triangles ADE and CDE have their vertices

at D, and their bases in the same line AE, they have the same altitude, and are to each other as their bases.

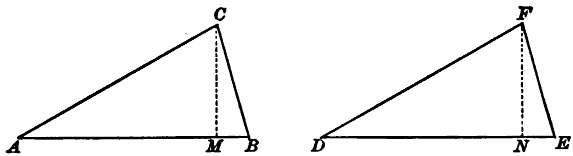
Hence $$\frac{\triangle ADE}{\triangle CDE} = \frac{AE}{CE}. \quad (1)$$

In like manner, $$\frac{\triangle ABE}{\triangle ADE} = \frac{BE}{DE}. \quad (2) \qquad \S\ 232, 3.$$

Multiplying (1) by (2), $$\frac{\triangle ABE}{\triangle CDE} = \frac{AE \times BE}{CE \times DE}. \qquad \text{Q. E. D.}$$

PROPOSITION XXV. THEOREM.

269. ***Two similar triangles are to each other as the squares of their homologous sides.***



Given—AC and DF, homologous sides of the similar triangles ABC and DEF.

To Prove— $\triangle ABC : \triangle DEF = \overline{AC}^2 : \overline{DF}^2$.

Dem.—Draw the altitudes CM and FN.

Since the triangles are similar,

$$AB : DE = AC : DF, \qquad \S\ 217$$

And $$\tfrac{1}{2}CM : \tfrac{1}{2}FN = AC : DF. \qquad \S\S\ 256, 129$$

Multiplying, $\triangle ABC : \triangle DEF = \overline{AC}^2 : \overline{DF}^2$. § 132. Q. E. D.

270. COR.—*Two similar triangles are to each other as the squares of any two homologous lines.*

Since $\triangle AMC$ and $\triangle DNF$ are similar, § 250

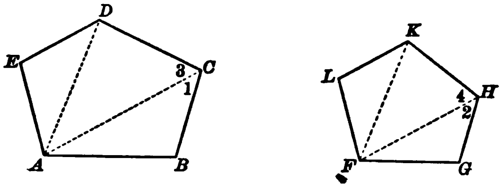
$$AC : DF = CM : FN = AM : DN, \qquad \S\ 217$$

And $$\overline{AC}^2 : \overline{DF}^2 = \overline{CM}^2 : \overline{FN}^2 = \overline{AM}^2 : \overline{DN}^2. \qquad \S\ 128$$

Hence $$\triangle ABC : \triangle DEF = \overline{CM}^2 : \overline{FN}^2 = \overline{AM}^2 : \overline{DN}^2. \qquad \S\ 131$$

PROPOSITION XXVI. THEOREM.

271. ***Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.***



Given—$\triangle ABC$ similar to $\triangle FGH$, $\triangle ACD$ to $\triangle FHK$, and $\triangle ADE$ to $\triangle FKL$.

To Prove—EB similar to LG.

Dem.—Since the corresponding triangles are similar,

$$\angle 1 = \angle 2.$$

$$\angle 3 = \angle 4. \qquad \S\ 217$$

Adding, $\angle C = \angle H$.

In like manner, $\angle D = \angle K$, and $\angle A = \angle F$.

Also $\angle B = \angle G$, and $\angle E = \angle L$.

Hence $ABCDE$ and $FGHKL$ are mutually equiangular.

Again, since the corresponding triangles are similar,

$$\frac{AB}{FG} = \frac{BC}{GH} = \left(\frac{AC}{FH}\right) = \frac{CD}{HK} = \left(\frac{AD}{FK}\right) = \frac{DE}{KL} = \frac{AE}{FL}. \qquad \S\ 217$$

Dropping the ratios in parentheses, the homologous sides are proportional.

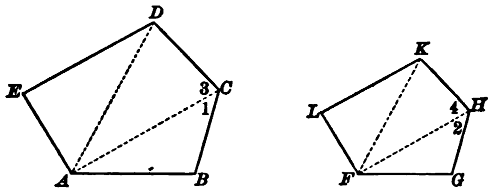
Therefore EB and LG are similar. Q. E. D.

EXERCISE.

233. What is the relation of two similar triangles whose altitudes are 6 and 9 feet respectively? Of two similar triangles whose bases are 8 and 12 chains respectively?

Proposition XXVII. Theorem.

272. Conversely—***Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.***



Given—*ABCDE* and *FGHKL* two similar polygons.

To Prove—$\triangle ABC$ similar to $\triangle FGH$, $\triangle ACD$ to $\triangle FHK$, and $\triangle ADE$ to $\triangle FKL$.

Dem.—Draw the homologous diagonals *AC*, *AD*, *FH*, and *FK*.

Since the polygons are similar,

$$\angle B = \angle G, \text{ and } \frac{AB}{FG} = \frac{BC}{GH}. \qquad \S\ 217$$

Hence $\triangle ABC$ and $\triangle FGH$ are similar. § 254

Since the polygons are similar,

$$\angle BCD = \angle GHK. \qquad (1) \qquad \S\ 217$$

And, since the triangles *ABC* and *FGH* are similar,

$$\angle 1 = \angle 2. \qquad (2) \qquad \S\ 217$$

Subtracting (2) from (1), $\angle 3 = \angle 4$.

Since the polygons are similar, and also $\triangle ABC$ is similar to $\triangle FGH$,

$$\text{Then} \qquad \frac{AC}{FH} = \left(\frac{BC}{GH}\right) = \frac{CD}{HK}. \qquad \S\ 217$$

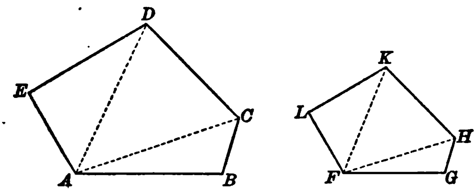
Dropping the ratio in parentheses,

$\triangle ADC$ and $\triangle FHK$ are similar. § 254

In like manner $\triangle ADE$ and $\triangle FKL$ are similar. Q. E. D.

PROPOSITION XXVIII. THEOREM.

273. ***The perimeters of two similar polygons are to each other as any two homologous sides.***



Given—P and P' the perimeters of the similar polygons EB and LG respectively.

To Prove— $\dfrac{P}{P'} = \dfrac{AB}{FG} = \dfrac{BC}{GH}$, etc.

Dem.—Since the polygons are similar,

$$\frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HK} = \frac{DE}{KL} = \frac{EA}{LF}. \qquad \S\ 217$$

Hence

$$\frac{AB + BC + CD + DE + EA}{FG + GH + HK + KL + LF} = \frac{AB}{FG} = \frac{BC}{GH}, \text{ etc.} \qquad \S\ 133$$

Or $\dfrac{P}{P'} = \dfrac{AB}{FG} = \dfrac{BC}{GH}$, etc. Q. E. D.

274. COR.—*The perimeters of any two similar polygons are to each other as any two homologous lines of the polygons.*

Since the polygons can be decomposed into the same number of similar triangles (§ 272),

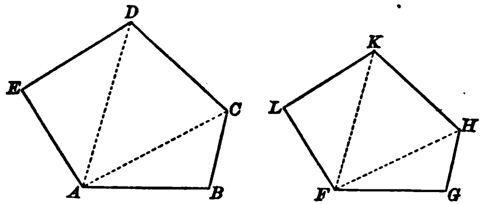
Then $\dfrac{AB}{FG} = \dfrac{AC}{FH} = \dfrac{AD}{FK}$. § 217

But $\dfrac{P}{P'} = \dfrac{AB}{FG}$. § 273

Hence $\dfrac{P}{P'} = \dfrac{AC}{FH} = \dfrac{AD}{FK}$. § 131. Q. E. D.

PROPOSITION XXIX. THEOREM.

275. *Two similar polygons are to each other as the squares of any two homologous sides.*



Given—P and P' the areas of the similar polygons EB and LG respectively.

To Prove— $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{FG}^2} = \frac{\overline{BC}^2}{\overline{GH}^2}$, etc.

Dem.—Since the polygons can be decomposed into the same number of similar triangles (§ 272), then

$$\frac{\triangle ABC}{\triangle FGH} = \left(\frac{\overline{AC}^2}{\overline{FH}^2}\right) = \frac{\triangle ACD}{\triangle FHK} = \left(\frac{\overline{AD}^2}{\overline{FK}^2}\right) = \frac{\triangle ADE}{\triangle FKL}. \quad \text{§ 269}$$

Dropping the ratios in parentheses, the corresponding triangles are proportional,

And $$\frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle FGH + \triangle FHK + \triangle FKL} = \frac{\triangle ABC}{\triangle FGH}. \quad \text{§ 133}$$

Therefore $$\frac{P}{P'} = \frac{\triangle ABC}{\triangle FGH} = \frac{\overline{AB}^2}{\overline{FG}^2} = \frac{\overline{BC}^2}{\overline{GH}^2}, \text{ etc.} \quad \text{Q. E. D.}$$

276. COR.—*Two similar polygons are to each other as the squares of any two homologous lines of the polygons.*

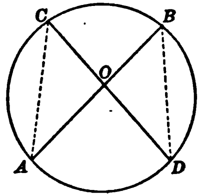
Since $$\frac{P}{P'} = \left(\frac{\triangle ABC}{\triangle FGH}\right) = \frac{\overline{AC}^2}{\overline{FH}^2}, \quad \text{§ 275}$$

Then $$\frac{P}{P'} = \frac{\overline{AC}^2}{\overline{FH}^2}. \quad \text{Ax. 1. Q. E. D.}$$

RELATION OF LINES IN A CIRCLE.

Proposition XXX. Theorem.

277. *If two chords intersect in a circle, their segments are reciprocally proportional.*



Given—AB and CD any two chords intersecting at O.

To Prove— $AO : CO = DO : BO$.

Dem.—Draw AC and BD.

Then in the triangles AOC and BOD

$\angle A = \angle D$, and $\angle C = \angle B$. § 179

Hence the triangles AOC and BOD are similar. § 251

Therefore $AO : CO = DO : BO$. § 217. Q. E. D.

278. Cor.—*The product of the segments of one chord is equal to the product of the segments of the other.*

From the proportion,

$$AO : CO = DO : BO,$$

Or $AO \times BO = CO \times DO$. § 119

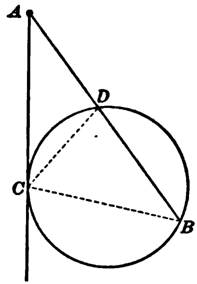
EXERCISES.

234. Prove that five times the square of the hypotenuse of a right triangle are equal to four times the sum of the squares of the medial lines from its extremities.

235. The area of a trapezoid is equal to the product of one of its legs into the distance from this leg to the middle point of the other leg of the trapezoid.

PROPOSITION XXXI. THEOREM.

279. ***If from a fixed point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and its external segment.***



Given—AB a secant and AC a tangent drawn from A to the circle CDB.

To Prove— $AB : AC = AC : AD$.

Dem.—Draw CB and CD.

Then in the triangles ACD and ACB

$\angle A$ is common, $\angle B = \angle ACD$. §§ 179, 185

Hence the triangles ACD and ACB are similar. § 251

Therefore $AB : AC = AC : AD$. § 217. Q. E. D.

280. COR.—*The square of the tangent is equal to the product of the whole secant and its external segment.*

From the proposition,

$$AB : AC = AC : AD,$$

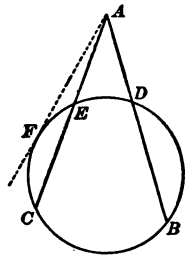
Or $\overline{AC}^2 = AB \times AD$. § 119

EXERCISE.

236. Required the area of an equilateral triangle whose sides are each 16 feet.

PROPOSITION XXXII. THEOREM.

281. *If from a fixed point without a circle two secants are drawn, the whole secants are reciprocally proportional to their external segments.*



Given—AB and AC two secants drawn from A to the circle CDB.

To Prove— $AB : AC = AE : AD$.

Dem.—Draw the tangent AF.

Then	$AC \times AE = \overline{AF}^2$,	§ 280
And	$AB \times AD = \overline{AF}^2$.	§ 280
Hence	$AB \times AD = AC \times AE$,	Ax. 1
Or	$AB : AC = AE : AD$.	§ 120. Q. E. D.

EXERCISES.

237. A trapezium is divided by its diagonals into four triangles proportional to one another.

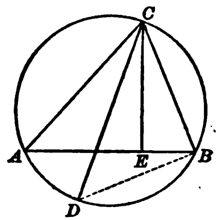
238. If three equal circles are tangent to one another, the area of the triangle formed by joining their centres is four times the area formed by joining their points of contact.

239. If two circles intersect, the common chord produced will bisect the common tangent.

240. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

PROPOSITION XXXIII. THEOREM.

282. ***In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side.***



Given—CE the altitude of the triangle ABC, and CD the diameter of the circumscribed circle.

To Prove— $AC \times BC = CD \times CE$.

Dem.—Draw DB, and $\angle DBC$ is a right angle. § 180

Then in the right triangles EAC and BDC

$\angle A = \angle D$. § 179

Hence $\triangle EAC$ and $\triangle BDC$ are similar, § 252

And $CA : CD = CE : CB$. § 217

Therefore $AC \times BC = CD \times CE$. § 119. Q. E. D.

EXERCISES.

241. Find the longest line that can be drawn on a floor whose length is 60 feet and width 40 feet.

242. Find the shortest distance from the lower corner of a room 40 feet long, 30 feet wide, and 12 feet high, to the opposite upper corner.

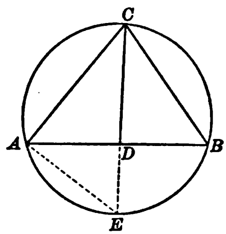
243. Find the shortest distance from the lower corner of a room 60 feet long, 40 feet wide, and 15 feet high, to the opposite upper corner, keeping on the surface of the room.

244. Find the area of a triangle whose sides are 8, 10, and 14 feet respectively.

245. A tree 120 feet high was broken off in a storm; the broken end remained on the stump, and the top struck 25 feet from the base: how high was the stump?

PROPOSITION XXXIV. THEOREM.

288. *In any triangle the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*



Given—CD the bisector of the angle C in the triangle ACB.

To Prove— $CA \times CB = AD \times BD + CD^2$.

Dem.—Circumscribe a circle about ABC, produce CD to E, and draw AE.

In the two triangles CAE and CDB,

By hypothesis, $\angle ACE = \angle BCD$,

And $\angle E = \angle B$. § 179

Hence the triangles CAE and CDB are similar, § 251

And $CA : CD = CE : CB$, § 217

Or $CA \times CB = CE \times CD$, § 119

$= (DE + CD)CD$,

$= DE \times CD + \overline{CD}^2$.

But $DE \times CD = AD \times DB$. § 278

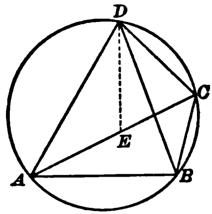
Substituting, $CA \times CB = AD \times DB + \overline{CD}^2$. Q. E. D.

EXERCISE.

246. The sides of a triangle are 12, 18, and 24 in. respectively: find the radius of the circumscribed circle.

PROPOSITION XXXV. THEOREM.

284. ***The product of the two diagonals of a quadrilateral inscribed in a circle is equal to the sum of the products of its opposite sides.***



Given—AC and DB the diagonals of the quadrilateral $ABCD$ inscribed in the circle ADC.

To Prove— $AC \times DB = AB \times DC + AD \times BC$.

Dem.—Draw DE, making $\angle ADE = \angle BDC$; then, adding to each $\angle EDB$, $\angle ADB = \angle EDC$.

In the triangles EDC and ADB, $\angle DCA = \angle DBA$. § 179

And, since $\angle EDC = \angle ADB$,

The triangles EDC and ADB are similar. § 251

Hence $DC : DB = EC : AB$, § 217

Or $DB \times EC = AB \times DC$. (1) § 119

Again, in the triangles ADE and BDC

$\angle DAC = \angle DBC$, § 179

And, by construction, $\angle ADE = \angle BDC$.

Then the triangles ADE and BDC are similar. § 251

Hence $AD : DB = AE : BC$, § 217

Or $DB \times AE = AD \times BC$. (2) § 119

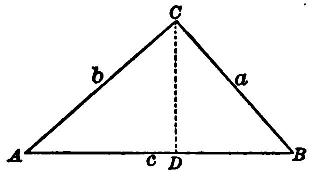
Adding (1) and (2), we have

$$DB \times (EC + AE) = AB \times DC + AD \times BC,$$

Or $DB \times AC = AB \times DC + AD \times BC$. Q. E. D.

PROPOSITION XXXVI. THEOREM.

285. *To find the area of a triangle when the three sides are given.*



Given—a, b, and c the three sides of the triangle ABC, and P its area.

Required—To find P in terms of a, b, and c.

Dem.—Draw the perpendicular CD.

Then $a^2 = b^2 + c^2 - 2c \times AD$, § 239

And $AD = \dfrac{b^2 + c^2 - a^2}{2c}.$

But $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2$, § 236

$$= (AC + AD)(AC - AD),$$

$$= \left(b + \frac{b^2 + c^2 - a^2}{2c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2c}\right),$$

$$= \frac{(b^2 + 2bc + c^2 - a^2)(a^2 - b^2 + 2bc - c^2)}{4c^2},$$

$$= \frac{[(b+c)^2 - a^2][a^2 - (b-c)^2]}{4c^2},$$

$$= \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4c^2}. \quad (1)$$

If we let $a + b + c = 2s$, then $b + c - a = 2(s - a)$, $a + c - b = 2(s - b)$, and $a + b - c = 2(s - c)$.

Substituting these values in (1), we have

$$\overline{CD}^2 = \frac{16s(s-a)(s-b)(s-c)}{4c^2},$$

And $\qquad CD = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{c}.$

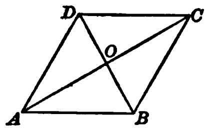
But $\qquad P = CD \times \frac{1}{2}c.$ $\qquad$ § 231.

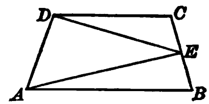
Hence $\qquad P = \sqrt{s(s-a)(s-b)(s-c)}.$ $\qquad$ Q. E. D.

EXERCISES.

ORIGINAL THEOREMS.

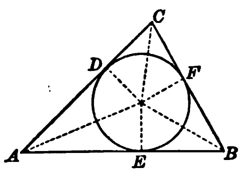
247. The area of a rhombus is equal to one half of the product of its diagonals (§ 85).

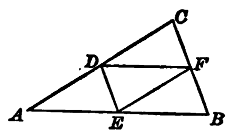




248. If the middle point of one of the non-parallel sides of a trapezoid is joined with the extremities of the opposite side, the triangle formed is one half of the trapezoid.

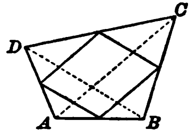
249. The area of a triangle is equal to one half the product of its perimeter by the radius of the inscribed circle.





250. If the points of bisection of the sides of a given triangle be joined, the triangle thus formed will be similar to the given triangle and equivalent to one fourth of it.

251. The four lines joining the middle points of the four sides of any quadrilateral form a parallelogram.

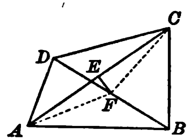


252. The perimeter of the parallelogram formed by joining the middle points of the four sides of any quadrilateral is equal to the sum of the diagonals of the quadrilateral.

253. The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side (§ 236).

254. If similar polygons are constructed upon the sides of a right triangle, the polygon on the hypotenuse is equal to the sum of the polygons on the other two sides.

255. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals, increased by four times the square of the line joining the middle points of the diagonals.



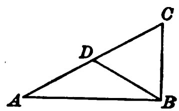
256. If one of the parallel sides of a trapezoid is double the other, each diagonal will cut off one third of the other.

257. If a line tangent to two circles cuts another line joining their centres, the segments of the latter will be to each other as the diameters of the circles.

258. If two circles are tangent to each other internally, the chords of the greater drawn from the point of tangency are divided proportionally by the circumference of the smaller.

259. If D is the middle point of the hypotenuse AC of the right triangle ABC, prove that

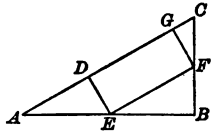
$$\overline{DB}^2 = \tfrac{1}{3}(\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2).$$



260. If the perpendicular from the vertex of a right angle upon the hypotenuse divides the hypotenuse into two parts which are to each other as 4 to 1, the longer leg of the triangle is double the shorter.

261. An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than, the sum of the squares of the other two sides.

262. If $DEFG$ is a rectangle inscribed in the right triangle ABC, prove DE a mean proportional between AD and GC.

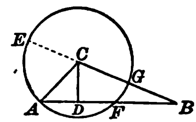


263. If two circles are tangent to each other, externally or internally, any two straight lines drawn through the point of contact and terminated both ways by the circumferences will be divided proportionally by them.

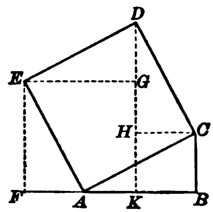
264. The altitude of an equilateral triangle is equal to three times the radius of the inscribed circle.

265. The sum of the squares of the segments of two chords which are perpendicular to each other is equal to the square of the diameter of the circle.

266. If from the vertical angle of a triangle a perpendicular be drawn to the base, the product of the sum and difference of the two sides is equal to the product of the sum and difference of the segments of the base (§ 281).



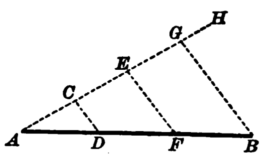
267. Prove Prop. VIII., Art. 235, by using the following diagram.



PROBLEMS OF CONSTRUCTION.

Proposition XXXVII. Problem.

286. *To divide a straight line into any number of equal parts.*



Given—AB any straight line.

Required—To divide AB into three equal parts.

Cons.—Draw the indefinite straight line AH, making any angle with AB.

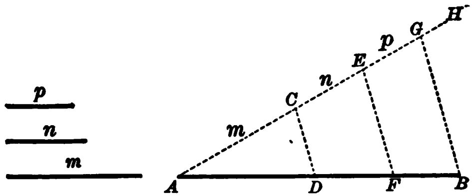
Take AC any length, and make CE and EG equal to AC.

Draw GB, then draw EF and CD parallel to GB.

Then $AD = DF = FB$. § 244. Q. E. F.

PROPOSITION XXXVIII. PROBLEM.

287. ***To divide a given straight line into parts proportional to any number of given lines.***



Given—AB any straight line.

Required—To divide AB proportional to m, n, and p.

Cons.—Draw the indefinite straight line AH, making any angle with AB.

Take $AC = m$, $CE = n$, and $EG = p$.

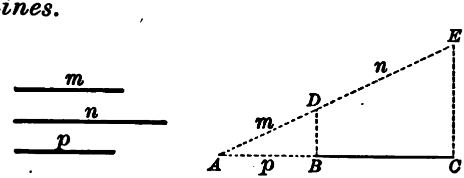
Draw GB, then draw EF and CD parallel to GB.

Then $$\frac{AD}{AC} = \frac{DF}{CE} = \left(\frac{AF}{AE}\right) = \frac{FB}{EG},$$ § 244

Or $$\frac{AD}{m} = \frac{DF}{n} = \frac{FB}{p}.$$ Q. E. F.

PROPOSITION XXXIX. PROBLEM.

288. ***To construct a fourth proportional to three given lines.***



Given—m, n, and p three lines.

Required—To construct a fourth proportional to m, n, and p.

Cons.—Take $AD = m$, $DE = n$, and $AB = p$.

Draw DB, then draw EC parallel to DB.

Then $AD : DE = AB : BC$, § 242

Or $m : n = p : BC$. § 113. Q. E. F.

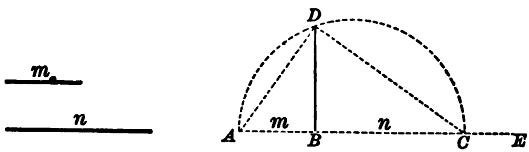
289. COR.—If AB is made equal to n, we shall have

$$m : n = n : BC.$$

BC is then a third proportional to m and n. § 112.

PROPOSITION XL. PROBLEM.

290. *To construct a mean proportional to two given lines.*



Given—m and n any two straight lines.

Required—To construct a mean proportional to m and n.

Cons.—Take $AB = m$, and $BC = n$.

Describe a semi-circumference on AC as a diameter, and erect BD perpendicular to AC.

Then BD is a mean proportional to m and n.

Proof.—For $AB : BD = BD : BC$, § 265

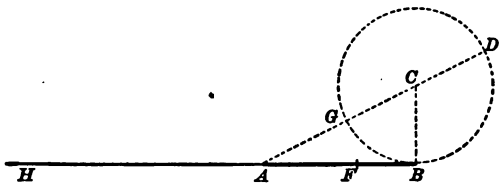
Or $m : BD = BD : n$. § 112. Q. E. F.

291. DEFINITION.—A straight line is divided into *Extreme and Mean Ratio* when one segment is a mean proportional between the whole line and the other segment.

When the point of division falls within the line, it is said to be divided *internally;* if the point falls without the line, it is divided *externally.*

PROPOSITION XLI. PROBLEM.

292. ***To divide a given straight line into extreme and mean ratio.***



Given—AB any straight line.

Required—To divide AB into extreme and mean ratio.

Cons.—Draw CB perpendicular to AB and equal to one half of it.

With C as a centre, and with a radius equal to one half of AB, describe the circumference GBD.

Draw AC, and produce it to D.

Take AF equal to AG, and AH equal to AD.

Then AB is divided into extreme and mean ratio *internally* at F, and *externally* at H.

Proof.—For $AD : AB = AB : AG$, § 279

Or $AD : AB = AB : AF$.

By division, $AD - AB : AB = AB - AF : AF$. § 126

But $AD - AB = AG$, or AF, and $AB - AF = BF$.

Hence $AF : AB = BF : AF$.

By inversion, $AB : AF = AF : BF$. § 291

Again, by composition,

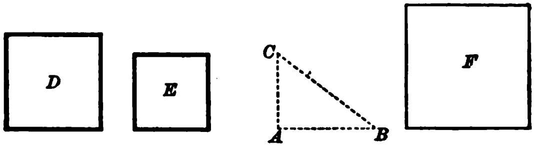
$$AD + AB : AD = AB + AF : AB. \quad § 125$$

But $AD + AB = BH$, $AH = AD$, and $AB + AF = AH$.

Hence $BH : AH = AH : AB$. Q. E. F.

PROPOSITION XLII. PROBLEM.

293. *To construct a square equivalent to the sum of two given squares.*



Given—D and E any two squares.

Required—To construct a square equivalent to the sum of D and E.

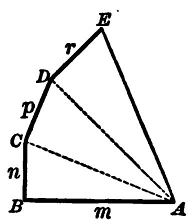
Cons.—Take AB equal to a side of D, erect AC perpendicular to AB and equal to a side of E, and draw CB.

Then the square F, described with a side equal to CB, will be equivalent to D plus E.

Proof.—For $\overline{CB}^2 \Leftrightarrow \overline{AB}^2 + \overline{AC}^2$, § 235

Or $F \Leftrightarrow D + E$. Q. E. F.

294. COR.—*To construct a square equivalent to the sum of any number of given squares.*



Given—m, n, p, and r the sides of four squares.

Required—To construct a square equivalent to the sum of the four given squares.

Cons.—Construct AB equal to m, and BC perpendicular to AB and equal to n, and draw AC.

At C erect CD perpendicular to AC and equal to p, and draw AD. At D erect DE perpendicular to AD and equal to r, and draw AE.

Then the square described on AE will be equivalent to the sum of the four given squares.

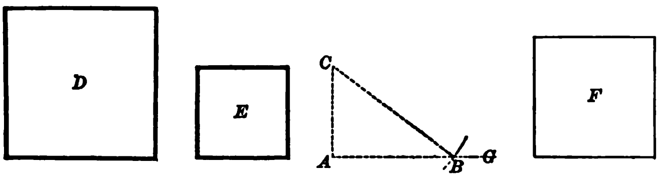
Proof.—For

$$\overline{AE}^2 \approx r^2 + \overline{AD}^2 \approx r^2 + p^2 + \overline{AC}^2 \approx r^2 + p^2 + n^2 + m^2.$$

§ 235. Q. E. F.

Proposition XLIII. Problem.

295. ***To construct a square equivalent to the difference of two given squares.***



Given—D and E any two squares.

Required—To construct a square equivalent to the difference of D and E.

Cons.—At A erect AC perpendicular to AG and equal to a side of E.

With C as a centre, and with a radius equal to a side of D, describe an arc at B.

Then the square F, described on AB, will be equal to the difference between D and E.

Proof.—For $\overline{AB}^2 = \overline{CB}^2 - \overline{AC}^2$, § 236

Or $F = D - E$. Q. E. F.

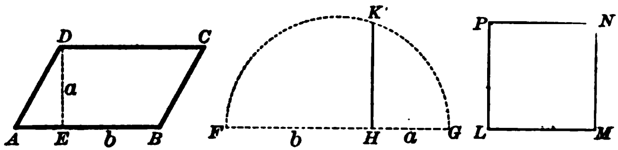
EXERCISES.

268. Construct a square equivalent to the difference between two squares whose sides are 5 inches and 9 inches respectively.

269. Construct a square equivalent to the difference between two squares whose sides are 12 inches and 8 inches respectively. Construct a square equivalent to their sum.

PROPOSITION XLIV. PROBLEM.

296. *To construct a square equivalent to a given parallelogram.*



Given—AC any parallelogram.

Required—To construct a square equivalent to AC.

Cons.—Construct HK a mean proportional to AB and DE. § 290

Then the square LN, described on HK, will be equal to the parallelogram AC.

Proof.—For, by construction,

$$FH : HK = HK : HG,$$

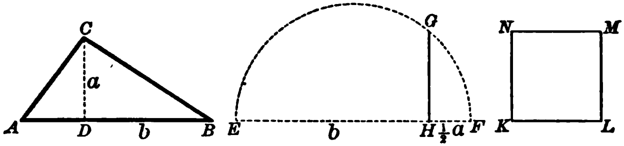
Or $AB : LM = LM : DE,$

And $\overline{LM}^2 = AB \times DE.$ § 119

Hence area $LMNP$ = area $ABCD$. Q. E. F.

PROPOSITION XLV. PROBLEM.

297. *To construct a square equivalent to a given triangle.*



Given—ABC any triangle.

Required—To construct a square equivalent to ABC.

Cons.—Construct HG a mean proportional to AB and $\frac{1}{2}DC$.

Then the square KM described on HG will be equivalent to the triangle ABC.

Proof.—For, by construction,

$$EH : GH = GH : HF,$$

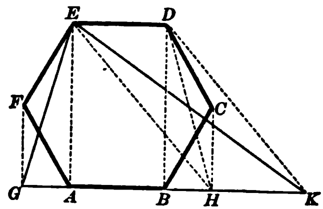
Or $$AB : KL = KL : \tfrac{1}{2}CD,$$

And $$\overline{KL}^2 = AB \times \tfrac{1}{2}CD. \qquad \S\ 119$$

Hence $$\text{area } KLMN = \text{area } ABC. \qquad \text{Q. E. F.}$$

PROPOSITION XLVI. PROBLEM.

298. ***To construct a triangle equivalent to a given polygon.***



Given—$ABCDEF$ any polygon.

Required—To construct a triangle equivalent to $ABCDEF$.

Cons.—Draw the diagonal BD, draw CH parallel to DB, and connect D and H.

Then $$\text{area } \triangle BCD = \text{area } \triangle BHD. \qquad \S\ 232, 4.$$

Hence $$\text{area } ABCDEF = \text{area } AHDEF.$$

Again, draw EH, draw DK parallel to EH, and connect E and K.

Then $$\text{area } \triangle EHD = \text{area } \triangle EHK. \qquad \S\ 232, 4.$$

Hence $$\text{area } AHDEF = \text{area } AKEF.$$

Draw the diagonal *AE*, draw *FG* parallel to *AE*, and connect *E* and *G*.

Then area $\triangle AEF =$ area $\triangle AEG$. § 232, 4.

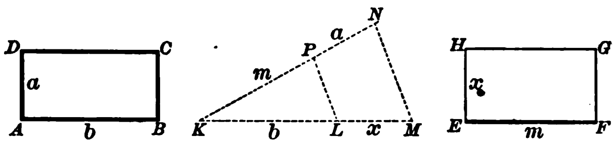
Hence area $AKEF =$ area GKE.

Therefore area $ABCDEF =$ area GKE. Ax. 1. Q. E. F.

299. Scholium.—By means of §§ 297 and 298 a square can be constructed equivalent to any given polygon.

Proposition XLVII. Problem.

300. ***On a given straight line as a base, to construct a rectangle equivalent to a given rectangle.***



Given—*ABCD* a rectangle, and *EF* any straight line.

Required—To construct on *EF* a rectangle equivalent to *ABCD*.

Cons.—Find a fourth proportional, *LM*, to *AB*, *AD*, and *EF*. § 288

Construct the rectangle *EFGH*, with *EF* and *EH* (= *LM*) as its adjacent sides. § 205

Then area $ABCD =$ area $EFGH$.

Proof.—For $KP : KL = PN : LM$, § 288

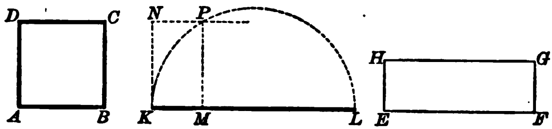
Or $EF : AB = AD : EH$,

And $AB \times AD = EF \times EH$. § 119

Hence area $ABCD =$ area $EFGH$. Q. E. F.

PROPOSITION XLVIII. PROBLEM.

301. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Given—$ABCD$ any square, and KL any line.

Required—To construct a rectangle equivalent to $ABCD$, having the sum of its base and altitude equal to KL.

Cons.—On KL as a diameter describe a semi-circumference.

Draw KN perpendicular to KL and equal to AB. Draw NP parallel to KL, intersecting the arc KPL at P, and draw PM perpendicular to KL.

Then the rectangle $EFGH$, constructed with ML as the base and MK as its altitude, is equivalent to the square $ABCD$.

Proof.—For $KM : PM = PM : LM$. § 265

Then $\overline{PM}^2 = KM \times LM$. § 119

Hence area $ABCD =$ area $EFGH$. Q. E. F.

SCHOLIUM.—By means of §§ 297 and 298 a rectangle can be constructed if its area is given in the form of any polygon.

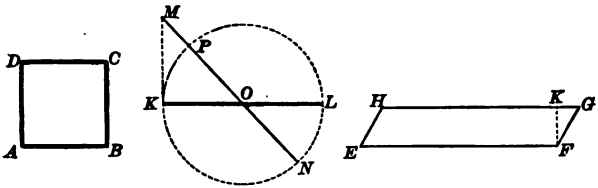
EXERCISES.

270. Construct a square equivalent to a tetragon.

271. Construct a square equivalent to a pentagon.

PROPOSITION XLIX. PROBLEM.

302. *To construct a parallelogram equivalent to a given square, having the difference of its base and altitude equal to a given line.*



Given—$ABCD$ any square, and KL any line.

Required—To construct a rectangle equivalent to $ABCD$, and having the difference of its base and altitude equal to KL.

Cons.—On KL as a diameter describe a circumference.

Draw KM perpendicular to KL and equal to AB. Draw MN through the centre O.

Then the parallelogram $EFGH$, constructed with MN as a base and MP as its altitude, is equivalent to the square $ABCD$.

Proof.—For $MN - MP = PN = KL$.

Hence $EF - FK = KL$.

Also $\overline{MK}^2 = MN \times MP$. § 279

Hence area $ABCD =$ area $EFGH$. Q. E. F.

EXERCISES.

By construction find x—

272. If $3:4 = 5:x$ (§ 288).

273. If $cx = ab$ (§ 288).

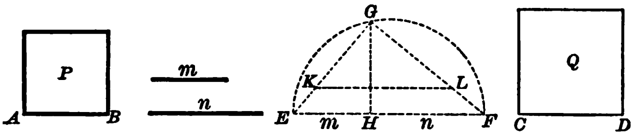
274. If $x^2 = 12$. If $x = \sqrt{8}$ (§ 290).

275. If $7:x = x:7-x$. If $x^2 + 9x = 81$ (§ 292).

276. If $x^2 = 3\sqrt{3}$. If $x^2 = 2\sqrt{5}$ (§ 290).

PROPOSITION L. PROBLEM.

303. *To construct a square having a given ratio to a given square.*



Given—P any square and any ratio $n : m$.

Required—To construct a square which shall have to P the ratio of $n : m$.

Cons.—On the line EF take EH equal to m and HF equal to n.

On EF construct the semi-circumference EGF.

Construct GH perpendicular to EF, and draw GE and GF.

On GE take GK equal to AB, and draw KL parallel to EF.

Then the square Q, constructed with its side equal to GL, will have to P the ratio of $n : m$.

Proof.—For $\angle EGF$ is a right angle. § 180

And, since GH is perpendicular to EF,

$$\frac{\overline{GF}^2}{\overline{GE}^2} = \frac{n}{m}. \qquad \S\ 264$$

But, since KL is parallel to EF,

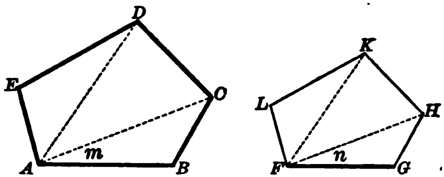
$$\frac{GL}{GK} = \frac{GF}{GE}. \qquad \S\ 242$$

Then $$\frac{\overline{GL}^2}{\overline{GK}^2} = \frac{\overline{GF}^2}{\overline{GE}^2} = \frac{n}{m}. \qquad \text{Ax. 2}$$

Hence $$\frac{\text{area } Q}{\text{area } P} = \frac{n}{m}. \qquad \S\ 227. \quad \text{Q. E. F.}$$

PROPOSITION LI. PROBLEM.

804. *To construct a polygon upon a given line similar to a given polygon.*



Given—*ABCDE* any polygon, and *FG* any line.

Required—To construct on *FG* a polygon similar to *ABCDE.*

Cons.—Draw the diagonals *AC* and *AD.*

Construct $\angle G$ equal to $\angle B$, and $\angle n$ equal to $\angle m$. § 195

Then the triangles *ABC* and *FGH* are similar. § 251

In like manner construct $\triangle FHK$ similar to $\triangle ACD$, and $\triangle FKL$ similar to $\triangle ADE$.

Then *FGHKL* will be similar to *ABCDE.* § 271. Q. E. F.

EXERCISES.

277. To construct a square equivalent to the sum of two given squares whose sides are 4 and 5 respectively (§ 293).

278. To construct a square equivalent to the difference of two given squares whose sides are 7 and 3 respectively (§ 295).

279. To divide a triangle into two equivalent parts by drawing a line from the vertex to the base (§ 232, Cor. III.).

280. To divide a triangle into two parts which shall be to each other as 3 to 4 by drawing a line from the vertex to the base.

281. To construct an isosceles triangle equivalent to a given triangle.

282. To construct a right triangle equivalent to a given triangle.

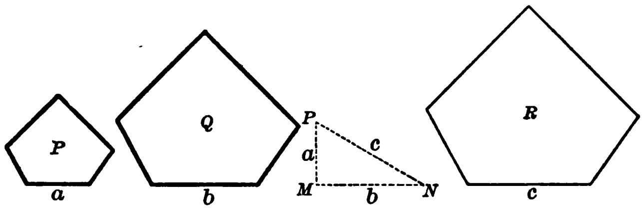
283. To construct a square five times a given square (§ 290).

284. To construct a square equivalent to a given trapezium (§ 299).

285. To divide the base of a triangle proportional to the other two sides (§ 246).

Proposition LII. Problem.

305. ***To construct a polygon similar to two given similar polygons and equivalent to their sum.***



Given—P and Q two similar polygons.

Required—To construct a polygon similar to P and equivalent to the sum of P and Q.

Cons.—Construct the right triangle PMN, with PM equal to a, the base of P, and MN equal to b, the base of Q.

Then on c, equal to PN, construct R, similar to P, § 304

And R is the required polygon.

Proof.—For $\frac{P}{R} = \frac{a^2}{c^2}$, and $\frac{Q}{R} = \frac{b^2}{c^2}$. § 275

Adding, $\frac{P + Q}{R} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$,

And $P + Q \Leftrightarrow R$. Q. E. F.

306. *In a similar manner a polygon may be constructed similar to two given similar polygons and equivalent to their difference.*

EXERCISES.

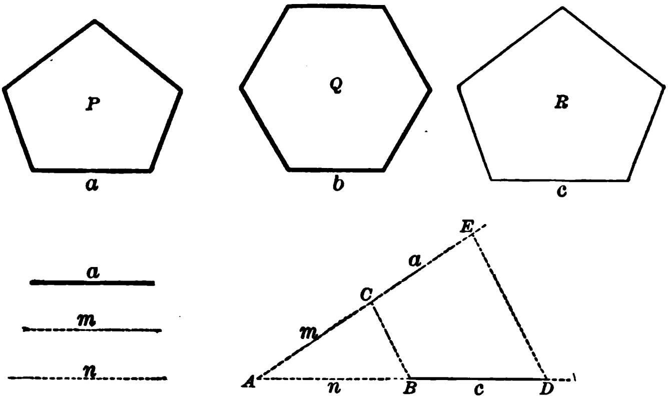
286. To bisect a given triangle by drawing a line parallel to the base.

287. To bisect a given triangle by drawing a line through a given point in one of its sides (§ 268).

288. To trisect a given triangle by drawing lines from the vertex to the base.

Proposition LIII. Problem.

307. ***To construct a polygon similar to a given polygon and equivalent to another polygon.***



Given—P and Q any two polygons.

Required—To construct a polygon similar to P and equivalent to Q.

Cons.—Construct squares equivalent to P and Q (§ 299), and let m and n denote the sides of the squares equal to P and Q respectively.

Then construct a fourth proportional to m, n, and a, the base of P. Let c be this fourth proportional.

Upon c, homologous to a, construct the polygon R similar to P. Then R will be the required polygon.

Proof.—For $m : n = a : c$,

And $m^2 : n^2 = a^2 : c^2$. § 128

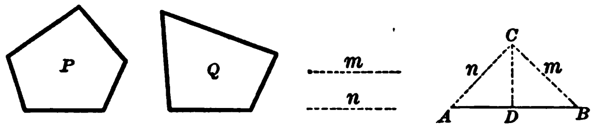
But $P \Leftrightarrow m^2$, and $Q \Leftrightarrow n^2$.

Then $P : Q = a^2 : c^2$; but $P : R = a^2 : c^2$. § 275

Hence $R \Leftrightarrow Q$. Q. E. F.

PROPOSITION LIV. PROBLEM.

308. *To find two straight lines which have the same ratio as the areas of two given polygons.*



Given—P and Q any two polygons.

Required—To construct two lines which have the same ratio as the areas of P and Q.

Cons.—Reduce P and Q to triangles (§ 298), and then to equivalent squares (§ 297).

Let m and n be the sides of squares equivalent to P and Q respectively.

Construct the right angle ACB (§ 192); take $CB = m$ and $CA = n$, and draw AB, and CD perpendicular to it.

Then $\quad$ area P : area $Q = DB : AD$.

Proof.—For $\quad \overline{BC}^2 : \overline{AC}^2 = DB : AD$. § 264

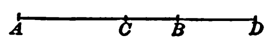
But $\quad \overline{BC}^2 = m^2 = P$, and $\overline{AC}^2 = n^2 = Q$.

Hence $\quad$ area P : area $Q = DB : AD$. Q. E. F.

HARMONIC PROPORTION.

309. A line is divided *Harmonically* when it is divided internally and externally in the same ratio.

Thus, if AB is divided internally at C and externally at D, so that



$$CA : CB = DA : DB,$$

then AB is divided harmonically.

Since this proportion may be written

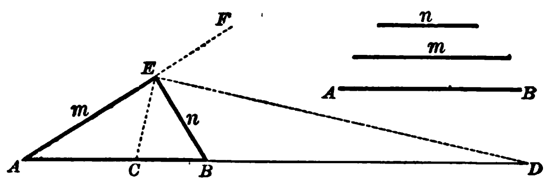
$$AC : AD = BC : BD,$$

that is, since the ratio of the distances of A from C and D is equal to the ratio of the distances of B from C and D, the line CD is divided harmonically at A and B.

The four points A, B, C, D are called *harmonic points*, and A and B are called *conjugate points*, as are also C and D.

PROPOSITION LV. PROBLEM.

310. ***To divide a given straight line harmonically in a given ratio.***



Given—AB any straight line.

Required—To divide AB harmonically in the ratio of m to n.

Cons.—Construct the triangle ABE having the three given lines AB, m, and n as sides. § 202

Bisect $\angle AEB$ by the line EC, and $\angle BEF$ by the line ED. § 193

Then AB is divided harmonically at C and D in the ratio of m to n.

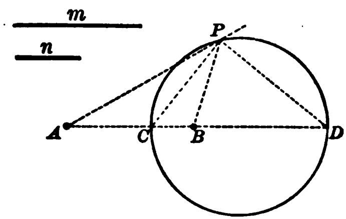
Proof.—For $CA : CB = m : n$, § 246

And $DA : DB = m : n$. § 248

Hence $CA : CB = DA : DB$. § 131. Q. E. F.

PROPOSITION LVI. PROBLEM.

311. ***To find the locus of all points whose distances from two given points are in a given ratio.***



Given—A and B any two points.

Required—To find the locus of all points whose distances from A and B are in the ratio of m to n.

Cons.—Divide AB internally at C and externally at D in the ratio of m to n. § 310

Upon CD as a diameter describe a circumference.

Then the required locus is the circumference CPD.

Proof.—For the angle CPD is a right angle, and the locus of P is the circumference of a circle described upon CD as a diameter. § 180. Q. E. F.

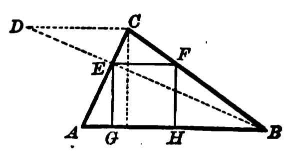
EXERCISES.

289. To construct a square having given the difference between the diagonal and the side.

290. To draw a line from the obtuse angle of a triangle to the base which shall be a mean proportional between the segments into which it divides the base.

291. Inscribe a circle in a given sector.

292. Inscribe a square in a given triangle.



293. Inscribe in a given triangle a rectangle whose sides are in a given ratio.

BOOK V.

REGULAR POLYGONS—MEASUREMENT OF THE CIRCLE.

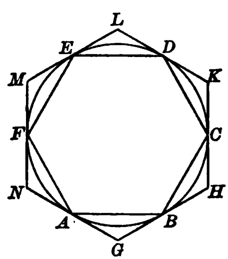
312. Definition.—A **Regular Polygon** is a polygon which is both equilateral and equiangular.

Proposition I. Theorem.

313. *If the circumference of a circle be divided into any number of equal parts,*

1. The chords joining the successive points of division form a regular inscribed polygon.

2. The tangents drawn at the points of division form a regular circumscribed polygon.



Given—ACE a circumference divided into any number of equal parts.

To Prove, 1st.—$ABCDEF$ a regular polygon.

Dem.—Since by hypothesis the arcs are equal,

Then chord $AB =$ chord $BC =$ chord CD, etc., § 156

And arc $BCDEF =$ arc $CDEFA$, etc.

Then $\angle A = \angle B = \angle C$, etc. § 179

Therefore the polygon $ABCDEF$ is regular. § 312

To Prove, 2d.—$GHKLMN$ a regular polygon.

Dem.—In the triangles AGB, BHC, CKD, etc.

$AB = BC = CD$, etc. § 156

And since arc $AB =$ arc $BC =$ arc CD, etc.,

$\angle GAB = \angle GBA = \angle HBC = \angle HCB$, etc. § 185

Hence $\triangle AGB = \triangle BHC = \triangle CKD$, etc. § 56

Whence $\angle G = \angle H = \angle K$, etc., § 45 (*a*)

And $AG = GB = BH = HC$, etc. § 66

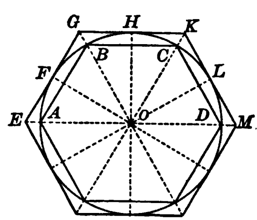
Then $GH = HK = KL$, etc.

Therefore the polygon $GHKLMN$ is regular. § 312. Q. E. D.

314. SCHOLIUM 1.—It is thus seen that if an inscribed polygon is given, a circumscribed polygon of the same number of sides can be formed by drawing tangents at the vertices of the given polygon. And if a circumscribed polygon is given, an inscribed polygon of the same number of sides can be formed by joining the consecutive points of tangency.

The following method is generally preferred:

315. SCHOLIUM 2.—1st. If $ABCD$ . . . be the inscribed polygon, bisect the arcs AB, BC, CD, etc. in the points F, H, L, etc., and draw the tangents EG, GK, KM, etc. at these points.



Then, since arc $FH =$ arc HL, etc.,

The polygon $EGKM$ . . . is regular. § 313

Since *OF* is perpendicular to both *AB* (§ 160) and *EG* (§ 169), the sides *AB* and *EG* are parallel.

For the same reason all the other sides of *ABCD*, etc. are parallel to the sides of *EGKM*, etc. respectively.

2d. If the circumscribed polygon *EGKM*, etc. is given,

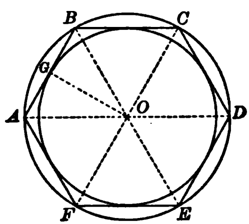
Draw *OE*, *OG*, *OK*, etc., and connect *AB*, *BC*, *CD*, etc., to obtain the inscribed polygon.

316. Scholium 3.—If the chords *AF*, *FB*, *BH*, etc. be drawn, a regular inscribed polygon of double the number of sides will be formed.

If tangents be drawn at *A*, *B*, *C*, etc., a regular circumscribed polygon of double the number of sides will be formed.

Proposition II. Theorem.

317. *A circle may be circumscribed about, or inscribed in, any regular polygon.*



Given—*ABCDEF* a regular polygon.

To Prove, 1st.—That a circle can be circumscribed about *ABCDEF*.

Dem.—A circumference can be described through *A*, *B*, and *C* (§ 162).

Let *O* be the centre of this circumference, and draw *OA*, *OB*, and *OC*.

Since $ABCDEF$ is a regular polygon,

$$\angle ABC = \angle BCD.$$

And, since $\triangle OBC$ is isosceles,

$$\angle OBC = \angle OCB. \qquad \S\, 63$$

Hence $\angle ABC - \angle OBC = \angle BCD - \angle OCB,$

Or $\angle ABO = \angle OCD.$ Ax. 2.

Also $OB = OC$, and $AB = CD$.

Hence $\triangle ABO = \triangle OCD,$ § 55

And $OA = OD.$

Therefore the circumference which passes through A, B, and C also passes through D. In the same manner it may be proved that the circumference which passes through A, B, C, and D also passes through E and F.

To Prove, 2d.—That a circle can be inscribed in $ABCDEF$.

Dem.—Since AB, BC, CD, etc. are equal chords of the circumscribed circle, they are equally distant from the centre O.

Then OG, etc. are all equal. § 163

Hence a circle described from O as a centre, and with a radius OG, will be inscribed in $ABCDEF$. Q. E. D.

DEFINITIONS.

318. The **Centre** of a regular polygon is the common centre, O, of the circumscribed and inscribed circles.

319. The **Radius** of a regular polygon is the radius, AO, of the circumscribed circle.

320. The **Apothem** of a regular polygon is the radius, OG, of the inscribed circle.

321. The *Angle at the centre* is the angle, AOF, included by the radii drawn to the extremities of any side.

322. Cor.—*The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

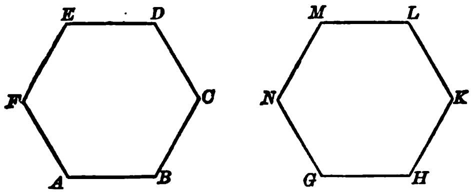
For $\angle AOB = \angle BOC = \angle COD$, etc., § 57

And $\angle AOB + \angle BOC$, etc. $=$ 4 right angles. § 19

Hence each angle at the centre equals four right angles divided by the number of sides.

Proposition III. Theorem.

323. ***Regular polygons of the same number of sides are similar.***



Given—*ABCDEF* and *GHKLMN* two regular polygons of the same number of sides.

To Prove—*ABCDEF* and *GHKLMN* similar.

Dem.—Since $AB = BC = CD$, etc., and $GH = HK = KL$, etc.,

$$\frac{AB}{GH} = \frac{BC}{HK} = \frac{CD}{KL}, \text{ etc.} \qquad \text{Ax. 2.}$$

Therefore the homologous sides of the polygons are proportional.

The sum of the angles of *ABCDEF* equals the sum of the angles of *GHKLMN*. § 98

Hence $\angle A = \angle G$, $\angle B = \angle H$, etc.

Therefore *ABCDEF* and *GHKLMN* are similar. § 217

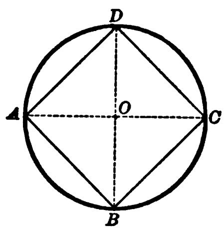
Q. E. D.

324. Cor.—*The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles; and their areas are to each other as the squares of these radii.*

For these radii are homologous lines of the similàr polygons. §§ 274, 276

Proposition IV. Problem.

325. *To inscribe a square in a given circle.*



Given—ABC any circle.

Required—To inscribe a square in AC.

Cons.—Draw the diameters AC and BD perpendicular to each other, and draw AB, BC, CD, and DA.

Then $ABCD$ is the required square.

Proof.—For arc $AB =$ arc $BC =$ arc $CD =$ arc DA. § 153

Then the chords AB, BC, etc. are equal, § 155

And the angles ABC, BCD, etc. are right angles. § 180

Therefore $ABCD$ is an inscribed square. § 313

326. Cor.—*The side of the inscribed square is to the radius as the square root of 2 is to 1.*

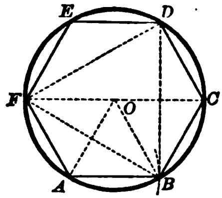
For $\overline{AB}^2 = \overline{AO}^2 + \overline{BO}^2 = 2\overline{AO}^2$, § 235

Or $AB = AO\sqrt{2}$. Ax. 2.

Hence $AB : AO = \sqrt{2} : 1$. § 120. Q. E. F.

PROPOSITION V. PROBLEM.

327. ***To inscribe a regular hexagon in a given circle.***



Given—*ACD* any circle.

Required—To inscribe a regular hexagon in *ACD.*

Cons.—Draw the radius *AO.*

With *A* as a centre and with *AO* as a radius, describe an arc at *B*, and draw *AB* and *BO.*

Then *AB* is a side of a regular inscribed hexagon.

Proof.—For the triangle *AOB* is equilateral, and $\angle AOB$ is one third of two right angles, or 60°.

Hence the arc *AB* is one sixth of a circumference,

And the chord *AB* is a side of a regular inscribed hexagon. § 313

Therefore, to inscribe a regular hexagon in a circle, apply the radius six times as a chord. Q. E. F.

328. COR. 1.—*The side of a regular inscribed hexagon is equal to the radius of the circle.*

329. COR. 2.—*By joining the alternate vertices of the regular hexagon an equilateral triangle is inscribed in the circle.*

330.—COR. 3.—*The side of an inscribed equilateral triangle is to the radius as the square root of 3 is to 1.*

For in the right triangle *FDC*,

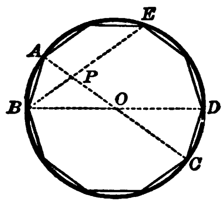
$$\overline{FD}^2 = \overline{FC}^2 - \overline{DC}^2 = 4\overline{CO}^2 - \overline{CO}^2 = 3\overline{CO}^2. \qquad \S\ 236$$

Or $$FD = CO\sqrt{3}.$$ Ax. 2.

Hence $$FD : CO = \sqrt{3} : 1.$$ § 120. Q. E. F.

PROPOSITION VI. PROBLEM.

331. *To inscribe a regular decagon in a given circle.*



Given—BD any circle.

Required—To inscribe a regular decagon in BD.

Cons.—Divide the radius AO in extreme and mean ratio (§ 292) at P.

Hence $$AO : PO = PO : PA. \qquad (1)$$

With A as a centre and with a radius equal to PO, describe an arc at B, and draw AB, BO, and BP.

Then AB is the side of a regular inscribed decagon.

Proof.—For in the triangles AOB and ABP, $\angle A$ is common.

But $AB = PO$; then proportion (1) gives

$$AO : AB = AB : PA.$$

Hence $\triangle AOB$ and $\triangle ABP$ are similar. § 254

Since $\triangle AOB$ is isosceles, $\triangle ABP$ is also isosceles, and $AB = BP = PO$,

And $$\angle O = \angle OBP.$$ § 63

Also $$\angle A = \angle APB = 2\angle O.$$ §§ 63, 49

Then, since $\angle A = \angle ABO$, the sum of the angles of the triangle AOB equals

$$\angle A + \angle ABO + \angle O = 2\angle O + 2\angle O + \angle O = 5\angle O.$$

Or $5\angle O = 180°$, and $\angle O = 36°$, or $\frac{1}{10}$ of a circumference.

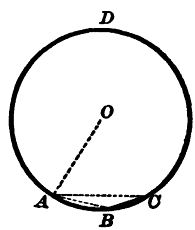
Therefore AB is the side of a regular inscribed decagon. § 313

Hence to inscribe a regular decagon in a circle divide the radius in extreme and mean ratio, and apply the greater segment as a chord ten times. Q. E. F.

332. COR.—*By joining the alternate vertices of the decagon, a regular pentagon will be inscribed in a circle.*

PROPOSITION VII. PROBLEM.

333. *To inscribe a regular pentedecagon in a given circle.*



Given—AD any circle.

Required—To inscribe a regular pentedecagon in AD.

Cons.—Let AC be the side of a regular inscribed hexagon; then arc AC is $\frac{1}{6}$ of the circumference.

Let AB be the side of a regular inscribed decagon; then arc AB is $\frac{1}{10}$ of the circumference.

Hence arc $AC -$ arc $AB =$ arc BC, is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$ of the circumference, and the chord BC is the side of a regular inscribed pentedecagon. Q. E. F.

334. COR. I.—*By bisecting the arcs subtended by the sides of any polygon, another polygon of double the number of sides can be inscribed in a circle.*

Thus, we can inscribe regular polygons—

From the *square* of 8, 16, 32, 64, etc.

From the *hexagon* of 12, 24, 48, 96, etc.

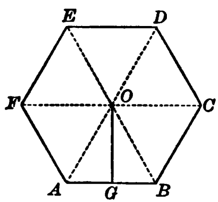
From the *decagon* of 20, 40, 80, 160, etc.

From the *pentedecagon* of 30, 60, 120, etc.

335. Until 1801 it was supposed that these were the only regular polygons that could be constructed by elementary geometry, which employs the straight line and circle only. But GAUSS, an eminent German mathematician, in his *Disquisitiones Arithmeticæ*, Lipsiæ, 1801, proved that it is possible to construct regular polygons of 17 sides, of 257 sides, and in general of any number of sides which can be expressed by 2^n+1, n being an integer and 2^n+1 a prime number.

PROPOSITION VIII. THEOREM.

336. *The area of a regular polygon is equal to the product of its perimeter by one half of its apothem.*



Given—AD a regular polygon, P its perimeter, and r the apothem.

To Prove—Area $AD = P \times \frac{1}{2}r$.

Dem.—Draw the radii OA, OB, etc., forming the triangles AOB, BOC, etc., with the common altitude r.

The area of $AOB = AB \times \frac{1}{2}r$,

" $BOC = BC \times \frac{1}{2}r$, etc. § 231

Adding, area $AD = (AB + BC +, \text{etc.}) \times \frac{1}{2}r$,

$= P \times \frac{1}{2}r$. Q. E. D.

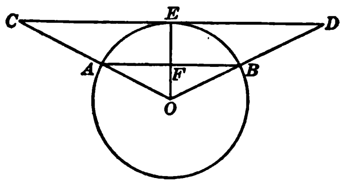
PROPOSITION IX. THEOREM.

337. *If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,*

1. The limit of its apothem is the radius.

2. The limit of its perimeter is the circumference.

3. The limit of its area is the area of the circle.



Given—AB and CD sides of two similar inscribed and circumscribed polygons.

To Prove, 1st.—That the limit of OF is OA.

Dem.—Draw the radius OE to the point of contact, then OE is perpendicular to AB (§ 315).

Then $$OA - OF < AF. \qquad \S\,47$$

If the number of sides of the inscribed polygon be increased indefinitely, the length of each side will be indefinitely diminished, and the limit of AB will be zero, and the limit of $\frac{1}{2}AB$, or AF, will also be zero.

Hence the limit of $OA - OF$ is zero.

Therefore the limit of OF is OA.

To Prove, 2d.—That the limit of the perimeters of the two polygons is the circumference of the circle.

Dem.—Let p and P denote the perimeters of two similar inscribed and circumscribed polygons respectively, and C the circumference.

Then $\qquad \dfrac{P}{p} = \dfrac{OE}{OF}.$ $\qquad$ § 324

Whence $\qquad \dfrac{P-p}{p} = \dfrac{OE-OF}{OF},$ $\qquad$ § 126

And $\qquad P-p = \dfrac{p}{OF}(OE-OF).$

If the number of sides of the polygons be indefinitely increased, we shall have two variables which are always equal; hence their limits are equal.

And $\qquad \lim.(P-p) = \lim.\dfrac{p}{OF}(OE-OF).$ $\qquad$ § 140

But $\quad \lim.(OE-OF) = 0.$ First part.

Whence $\quad \lim.(P-p) = 0,$

Or $\qquad \lim. P = \lim. p.$ $\qquad$ § 142

But the circumference of the circle lies between the perimeters of the two polygons.

Therefore $\qquad \lim. P = C = \lim. p.$

To Prove, 3d.—That the limit of the areas of the two polygons is the area of the circle.

Dem.—Let s and S denote the areas of two similar inscribed and circumscribed polygons respectively, and K the area of the circle.

The difference between the triangles COD and AOB is the trapezoid $ABDC$, whose area is

$\frac{1}{2}(CD + AB) \times EF.$ $\qquad$ § 233

Hence the difference between the areas of the polygons is

$$S - s = \tfrac{1}{2}(P+p)EF.$$

If the number of the sides of the polygon be indefinitely increased, we shall have two variables which are always equal; hence their limits are equal.

And $\lim. (S - s) = \lim. \frac{1}{2}(P + p)EF.$ § 140

But $\lim. EF = 0.$ § 139

Whence $\lim. (S - s) = 0,$

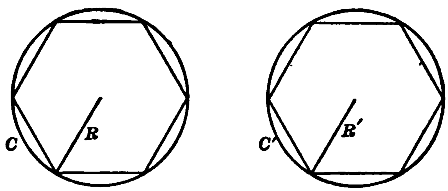
Or $\lim. S = \lim. s.$ § 142

But the area of the circle is evidently greater than that of the inscribed polygon, and less than the area of the circumscribed polygon.

Therefore $\lim. S = K = \lim. s.$ Q. E. D.

Proposition X. Theorem.

338. ***The circumferences of circles are to each other as their radii.***



Given—C and C' the circumferences of two circles, and R and R' their radii.

To Prove— $\frac{C}{C'} = \frac{R}{R'}.$

Dem.—Inscribe in the circles regular polygons of the same number of sides, and denote the perimeters by P and P'.

Then $\frac{P}{P'} = \frac{R}{R'}.$ § 324

Whence $PR' = P'R.$ § 119

If, now, the number of sides of each polygon be indefinitely increased, P will approach C as its limit, and P' will approach C' as its limit (§ 337).

Then, by the theory of Limits,

$$\text{lim. } PR' = \text{lim. } P'R. \qquad \S\,140$$

Or $$CR' = C'R, \qquad \S\,145$$

And $$\frac{C}{C'} = \frac{R}{R'}. \qquad (1) \quad \S\,120. \text{ Q.E.D.}$$

339. Cor. 1.—*The circumferences of circles are to each other as their diameters.*

If in equation (1) we multiply both terms of the ratio $\frac{R}{R'}$ by 2, we have

$$\frac{C}{C'} = \frac{2R}{2R'}, \quad \text{or } \frac{C}{C'} = \frac{D}{D'}. \quad (2)$$

340. Cor. 2.—*Similar arcs are to each other as their radii.*

By § 218, similar arcs are arcs which subtend equal angles at the centre, and are therefore like parts of their respective circumferences.

Let n be any constant quantity.

Since $$C : C' = R : R', \qquad \S\,338$$

Then $$\frac{C}{n} : \frac{C'}{n} = R : R'. \qquad \S\,129$$

But $\frac{C}{n}$ and $\frac{C'}{n}$ are similar arcs.

Hence similar arcs are to each other as their radii.

341. Cor. 3.—*The ratio of the circumference of a circle to its diameter is constant.*

From the proportion (2) we have

$$\frac{C}{D} = \frac{C'}{D'}.$$

This constant ratio is denoted by π.

Hence $$\frac{C}{D} = \pi. \qquad (3)$$

342. COR. 4.—*The circumference of a circle is πD, or $2\pi R$.*

For, from equation (3), we have

$$\frac{C}{D} = \pi, \text{ or } C = \pi D,$$

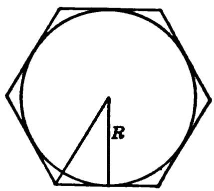
And, since $D = 2R$,

Substituting, $C = 2\pi R$.

NOTE.—The numerical value of π can be obtained only approximately. For practical purposes 3.1416 is used.

PROPOSITION XI. THEOREM.

343. *The area of a circle is equal to half the product of its circumference by its radius.*



Given—S the area, C the circumference, and R the radius of a circle.

To Prove— $S = \frac{1}{2}C \times R$.

Dem.—Circumscribe a regular polygon about the circle, and denote its area by A and its perimeter by P.

Then $A = \frac{1}{2}P \times R$. § 336

If, now, the number of sides of the polygon be indefinitely increased, A will approach S as its limit, and P will approach C as its limit (§ 337).

Then, by the theory of Limits,

$$\lim. A = \lim. \tfrac{1}{2}P \times R. \quad \text{§ 140}$$

Or $S = \frac{1}{2}C \times R$. (1) § 145. Q. E. D.

344. COR. 1.—*The area of a circle is πR^2, or $\frac{1}{4}\pi D^2$.*

For $C = 2\pi R$. § 342

Substituting in (1), $S = \frac{1}{2} \times 2\pi R \times R$,

$$S = \pi R^2.$$

Also $S = \pi(\frac{1}{2}D)^2 = \frac{1}{4}\pi D^2$.

345. COR. 2.—*The areas of circles are to each other as the squares of their radii or as the squares of their diameters.*

Let S and S' denote the areas of two circles, R and R' their radii, and D and D' their diameters.

Then $\dfrac{S}{S'} = \dfrac{\pi R^2}{\pi R'^2} = \dfrac{R^2}{R'^2}$,

And $\dfrac{S}{S'} = \dfrac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2} = \dfrac{D^2}{D'^2}$.

346. COR. 3.—*The area of a sector equals one half the product of its arc and radius.*

For denote the arc of the sector by c, and the area of the sector by s.

Let n be any constant quantity.

Since $S = \frac{1}{2}C \times R$, § 343

Then $\dfrac{S}{n} = \frac{1}{2}\dfrac{C}{n} \times R$. Ax. 2.

But, since s and c are like parts of S and C,

Then $\dfrac{S}{n} = s$, and $\dfrac{C}{n} = c$.

Whence $s = \frac{1}{2}c \times R$.

EXERCISES.

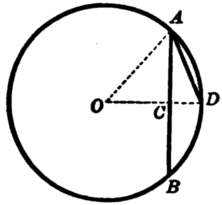
294. The diameters of two circles are 16 and 24 respectively. What is the ratio of their areas?

295. Find the area of a circle whose diameter is 12 inches.

296. The area of a circle is four times the area of the circle described upon its radius as a diameter.

PROPOSITION XII. PROBLEM.

347. ***To find the value of the chord of one half an arc, in terms of the chord of the whole arc, when the radius of the circle is 1.***



Given—AB the chord of the arc AB, and AD the chord of one half the arc AB, in a circle whose radius is **1**.

Required—To find the value of AD in terms of AB.

Cons.—Draw the radius OD perpendicular to AB, then OD bisects the chord AB and also its arc (§ 160), and draw AO.

Then in the triangle AOD,

$$\overline{AD}^2 = \overline{AO}^2 + \overline{DO}^2 - 2DO \times CO. \qquad \text{§ 239}$$

But when the radius is 1,

$$\overline{AD}^2 = 2 - 2CO. \qquad (1)$$

In the right triangle AOC,

$$\overline{CO}^2 = \overline{AO}^2 - \overline{AC}^2. \qquad \text{§ 236}$$

But, since $AO = 1$, and $AC = \frac{1}{2}AB$,

Then $\overline{CO}^2 = 1 - \frac{1}{4}\overline{AB}^2.$

Clearing of fractions,

$$4\overline{CO}^2 = 4 - \overline{AB}^2,$$

Or $2CO = \sqrt{4 - \overline{AB}^2}.$

Substituting the value of $2CO$ in (1),

$$\overline{AD}^2 = 2 - \sqrt{4 - \overline{AB}^2},$$

And $AD = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}.$ Q. E. D.

348. COR. 1.—*If we denote AB by S and AD by s, we have the*

General Formula $s = \sqrt{2 - \sqrt{4 - S^2}}$.

349. COR. 2.—*If the radius of the circle is R, the general formula is*

$$s = \sqrt{R(2R - \sqrt{4R^2 - S^2})}.$$

PROPOSITION XIII. PROBLEM.

350. ***To compute the numerical value of π, approximately.***

Since $C = 2\pi R$, § 342

When $R = 1, \quad \pi = \frac{1}{2}C$.

Hence the value of π is one half of the circumference of a circle whose radius is 1.

By beginning with a regular inscribed hexagon the side of which is equal to the radius of the circle, and finding the perimeters of the successive inscribed polygons of double the number of sides, by using the general formula of § 348,

$$s = \sqrt{2 - \sqrt{4 - S^2}},$$

we can find the approximate value of the circumference of a circle whose radius is 1.

Denote one side of a regular inscribed

Hexagon by s_6,

Dodecagon by s_{12},

Polygon of 24 sides by s_{24}, etc.

Then, when $R = 1, \quad s_6 = 1$. § 328

Substituting this value of s_6 in the general formula,

$$s_{12} = \sqrt{2 - \sqrt{4 - 1}},$$

Or $$s_{12} = \sqrt{2 - \sqrt{3}};$$

And $$s_{24} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{3})}},$$

Or $$s_{24} = \sqrt{2 - \sqrt{2 + \sqrt{3}}};$$

And $$s_{48} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{2 + \sqrt{3}})}},$$

Or $$s_{48} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}};$$

And doubling the number of sides *seven* times, in which case the polygon will have 768 sides,

$$s_{768} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}}}}$$
$$= .00818121.$$

If there are 768 sides in the perimeter, there are 384 sides in the semi-perimeter.

Hence the semi-perimeter is

$$384 \times s_{768} = 384 \times .00818121 = 3.14158464.$$

We may regard 3.14158464 as the approximate value of the semi-circumference of a circle whose radius is 1, and therefore the approximate value of π. Q. E. D.

351. SCHOLIUM.—Archimedes, who was born 287 B. C., was the first to assign an approximate value to π. By inscribing and circumscribing regular polygons of the same number of sides, and then doubling the number of sides, he proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$, or, in decimals, between 3.1428 and 3.1408.

Clausen and Dase, of Germany, about A.D. 1846, computed the value of π independently of each other, and their results agreed to the two hundredth decimal place. Others have carried the value to over seven hundred decimal places.

The value to fifteen decimal places is

$$\pi = 3.141592653589793.$$

EXERCISES.

PROBLEMS FOR CONSTRUCTION.

297. To construct a circle equivalent to the sum of two given circles.

298. To construct a circle equivalent to the difference of two given circles.

299. To construct a circle double a given circle.

300. To construct a circle three times a given circle.

301. To construct a circle four times a given circle.

302. To divide a given circle into four equivalent parts by concentric circumferences.

303. To construct a square in a sector equal to one fourth of a circle.

304. To construct a square in a semicircle.

305. To circumscribe about a circle a regular hexagon. A regular decagon.

306. To construct a regular pentagon, having given its radius.

307. To construct a regular hexagon, having given one side.

308. To inscribe a regular hexagon in a given equilateral triangle.

309. To inscribe a regular octagon in a given square.

310. To inscribe three equal circles in a given equilateral triangle, tangent to each other and to the sides of the triangle.

311. To inscribe three equal circles in a given circle, tangent to each other and to the given circle.

312. To construct a circle which shall touch one side of a triangle and the other two sides produced.

ORIGINAL THEOREMS.

313. The equilateral triangle circumscribed about a circle is four times the equilateral triangle inscribed in the circle.

314. The square circumscribed about a circle is double the square inscribed in the circle.

315. The regular hexagon circumscribed about a circle is four thirds of the area of the regular hexagon inscribed in the circle.

316. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

317. The square inscribed in a semicircle is equivalent to two fifths of the square inscribed in the entire circle.

318. The square inscribed in a quadrant is five eighths of the square inscribed in a semicircle.

319. The radius of any inscribed regular polygon is a mean proportional between its apothem and the radius of a similar circumscribed regular polygon.

320. The area of the ring included between two concentric circles is equal to the area of a circle whose diameter is a chord of the outer circle and a tangent to the inner.

321. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

322. An equiangular polygon inscribed in a circle is regular if the number of its sides is odd.

323. An equilateral polygon inscribed in a circle is regular.

324. An equiangular polygon circumscribed about a circle is regular.

325. The area of an inscribed regular dodecagon is equal to three times the square of the radius. Prove geometrically.

If R represents the radius, a the apothem, and s the side of a regular inscribed polygon, prove that—

326. In an equilateral triangle,

$$s = R\sqrt{3}, \text{ and } a = \tfrac{1}{2}R.$$

327. In a square,

$$s = R\sqrt{2}, \text{ and } a = \tfrac{1}{2}R\sqrt{2}.$$

328. In a regular hexagon,

$$S = R, \text{ and } a = \tfrac{1}{2}R\sqrt{3}.$$

329. In a regular decagon,

$$s = \tfrac{1}{2}R(\sqrt{5} - 1), \text{ and } a = \tfrac{1}{4}R\sqrt{10 + 2\sqrt{5}}.$$

330. In a regular pentagon,

$$s = \tfrac{1}{2}R\sqrt{10 - 2\sqrt{5}}, \text{ and } a = \tfrac{1}{4}R\sqrt{6 + 2\sqrt{5}}.$$

331. In a regular octagon,

$$s = R\sqrt{2 - \sqrt{2}}, \text{ and } a = \tfrac{1}{2}R\sqrt{2 + \sqrt{2}}.$$

332. In a regular dodecagon,

$$s = R\sqrt{2 - \sqrt{3}}, \text{ and } a = \tfrac{1}{2}R\sqrt{2 + \sqrt{3}}.$$

333. The square of the side of a regular inscribed pentagon, minus the square of the side of a regular inscribed decagon, is equal to the square of the radius.

If R represents the radius of a circle, prove that—

334. The area of an inscribed equilateral triangle is $\tfrac{3}{4}R^2\sqrt{3}$.

335. The area of an inscribed square is $2R^2$.

336. The area of a regular inscribed hexagon is $\frac{3}{2}R^2\sqrt{3}$.

337. The area of a regular inscribed octagon is $2R^2\sqrt{2}$.

338. The area of a regular inscribed dodecagon is $3R^2$.

339. The area of a regular inscribed pentagon is $\frac{5}{8}R^2\sqrt{10+2\sqrt{5}}$.

340. If a circle be circumscribed about a right triangle, and on each of its legs as a diameter a semicircle be described exterior to the triangle, the sum of the areas of the semicircles exterior to the circumscribed circle is equal to the area of the triangle.

PRACTICAL EXAMPLES.

341. Find the area of a circle whose diameter is 10 inches.

342. Find the area of a circle whose circumference is 60 feet.

343. Find the diameter and circumference of a circle whose area is 254.4696 sq. ft.

344. The radius of a circle is 12 in.; what is the radius of a circle $5\frac{4}{9}$ times as large?

345. The radii of three circles are 3, 4, and 5 respectively; what is the radius of a circle equivalent to their sum?

346. What is the radius of a circle whose area is one ninth of the area of a circle whose radius is 12 inches?

347. What is the area of a sector whose angle is 40°, in a circle whose diameter is 24 feet?

348. Find the length of a chord in a circle 10 inches in diameter, that is 4 inches from the centre.

349. Find the area of a square inscribed in a circle whose radius is 12 inches.

350. Find the area of a regular pentagon inscribed in a circle whose radius is 16 inches.

351. Find the area of a regular hexagon inscribed in a circle whose radius is 8 inches.

352. Find the area of a regular octagon inscribed in a circle whose radius is 10 inches.

353. How many degrees in an arc whose length is equal to the length of the radius?

354. Find the radius of a circle equivalent to an equilateral triangle whose side is 20 inches.

355. If a semicircle be described on each of the three sides of a right triangle, the semicircle on the hypotenuse is equal to the sum of the semicircles on the other two sides.

SUPPLEMENT TO PLANE GEOMETRY.

MAXIMA AND MINIMA.

352. A **Maximum** magnitude, among quantities of the same kind, is that which is the greatest.

353. A **Minimum** magnitude, among quantities of the same kind, is that which is the least.

Thus, the maximum straight line inscribed in a circle is the diameter; and the minimum straight line that can be drawn from a point to a straight line is the perpendicular.

In general a maximum or a minimum value of a mathematical expression is the value that is greater or less than any of the values which immediately precede or follow it.

Hence a mathematical expression may have two or more maxima or two or more minima values.

Thus, in $y = x^4 - 16x^3 + 88x^2 - 192x + 160$,

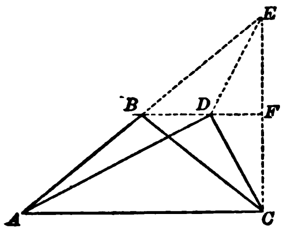
If $x = 1$, $y = 41$;
$x = 2$, $y = 16$, a minimum;
$x = 3$, $y = 25$;
$x = 4$, $y = 32$, a maximum;
$x = 5$, $y = 25$,
$x = 6$, $y = 16$, a minimum;
$x = 7$, $y = 41$.

But in elementary geometry, which treats only of the line and the circle, there can be but one maximum or one minimum value.

354. **Isoperimetric** figures are those which have equal perimeters.

PROPOSITION I. THEOREM.

355. ***Of all triangles having the same base and areas, that which is isosceles has the minimum perimeter.***



Given—ABC an isosceles triangle, and ADC any other triangle having the same base and equal area.

To Prove— $(AB + BC) < (AD + DC)$.

Dem.—Produce AB to E, making $BE = BC$, and draw EC.

Then $\angle ACE$ is a right angle, since it can be inscribed in a semicircle whose centre is B. § 180

Since the triangles ABC and ADC have equal bases and equal areas, their altitudes are equal, and BD is parallel to AC and perpendicular to EC at its middle point, §§ 36, 64

And	$DC = DE.$	§ 22
Now,	$(AB + BE) < (AD + DE).$	Ax. 5.
But	$BE = BC$, and $DE = DC.$	
Hence	$(AB + BC) < (AD + DC).$	Q. E. D.

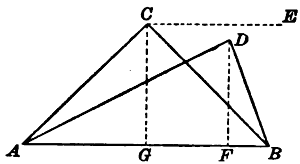
356. COR.—*Of all triangles having the same area, that which is equilateral has the minimum perimeter.*

For if the triangle with minimum perimeter is not isosceles when any side is taken as the base, its perimeter would be diminished by making it isosceles.

Therefore the triangle with minimum perimeter is equilateral.

PROPOSITION II. THEOREM.

357. ***Of isoperimetric triangles having the same base, that which is isosceles is the maximum.***



Given—ABC and ABD two isoperimetric triangles having the same base, AB, with ABC isosceles.

To Prove— Area $ABC >$ area ABD.

Dem.—Draw the altitudes CG and DF.

Then $CG > DF$; for if $CG = DF$,

$(AD + DB) > (AC + CB)$. § 355

And if $CG < DF$, then very much more would

$(AD + DB) > (AC + CB)$.

But both of these conclusions are contrary to the hypothesis.

Hence $CG > DF$. (1)

Multiplying both members of (1) by $\frac{1}{2}AB$,

We have $(\frac{1}{2}AB \times CG) > (\frac{1}{2}AB \times DF)$.

Whence area $ABC >$ area ABD. § 231. Q. E. D.

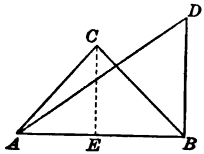
358. COR.—*Of all isoperimetric triangles, that which is equilateral is the maximum.*

For if the maximum triangle is not isosceles when any side is taken as the base, its area would be increased by making it isosceles. § 357

Therefore the maximum triangle is equilateral.

PROPOSITION III. THEOREM.

359. *Of all triangles formed with the same two given sides, that in which these sides are perpendicular is the maximum.*



Given—The triangles ABC and ABD, with $BC = BD$, and BD perpendicular to AB.

To Prove— Area $ABD >$ area ABC.

Dem.—Draw CE perpendicular to AB.

Then $CB > CE$. § 28

Hence $DB > CE$. (1)

Multiplying both members of (1) by $\frac{1}{2}AB$,

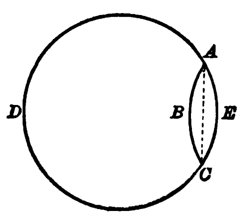
We have $(\frac{1}{2}AB \times DB) > (\frac{1}{2}AB \times CE)$.

Whence area $ABD >$ area ABC. § 231. Q. E. D.

PROPOSITION IV. THEOREM.

360. *Of all isoperimetric plane figures, the circle is the maximum.*

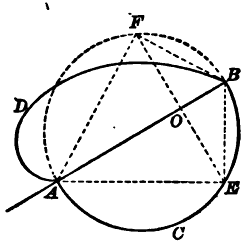
1st. The maximum figure must be convex.



If $ABCD$ be a concave figure, and the re-entrant portion, ABC, be revolved about the line AC into the position AEC, the figure $AECD$ will have the same perimeter as $ABCD$, but a greater area.

Hence the maximum figure must be convex.

2d. The straight line which bisects the perimeter of a maximum plane figure bisects its area also.



Let AB bisect the perimeter of $ACBDA$, then will AB bisect its area also.

For if ACB and ADB are unequal, suppose $ACB > ADB$; then if ACB be revolved about the line AB to the position of AFB, the area of $ACBFA$ would be greater than $ACBDA$. But by hypothesis $ACBDA$ is a maximum.

Hence ACB and ADB cannot be unequal.

Therefore the area is bisected.

3d. The figure $ACBFA$ is a circle.

Draw EF perpendicular to AB from any point in the arc ACB.

Then from the revolution of ACB about AB as an axis,

$$EO = FO,$$

And $\triangle AEB = \triangle AFB$. § 232

Now, the angles AFB and AEB must be right angles, for, if they are not, the areas of the triangles AEB and AFB

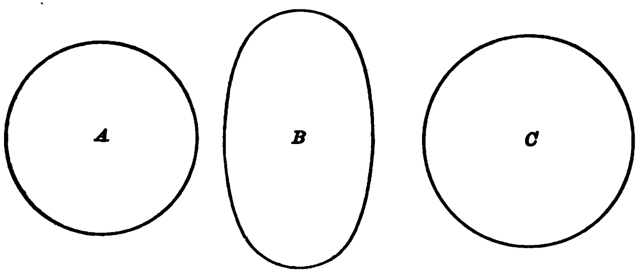
could be increased without increasing the chords AF, FB, AE, EB. § 359

In this case the whole figure would have its area increased without changing the perimeter, and $ACBFA$ would not be a maximum.

Therefore the angles AFB and AEB are right angles, and ACB and AFB are semicircumferences, and the whole figure is a circle. Q. E. D.

Proposition V. Theorem.

361. ***Of all plane figures containing the same area, the circle has the minimum perimeter.***



Given—A a circle, and B any other figure having the same area as A.

To Prove—Perimeter of $A <$ perimeter of B.

Dem.—Draw the circle C, having the same perimeter as the figure B.

Then $B < C$. § 360

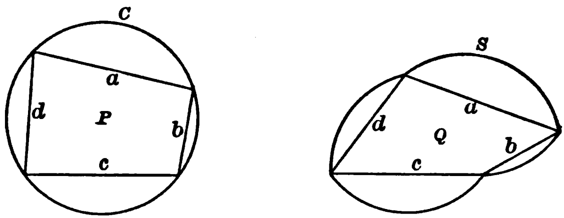
And, since the area of $A =$ area of B, $A < C$.

Now, of two circles, that which has the less area has the less perimeter. §§ 338, 345

Therefore the perimeter of A is less than the perimeter of C, or less than that of B. Q. E. D.

PROPOSITION VI. THEOREM.

362. ***Of all the polygons constructed with the same given sides, that is the maximum which can be inscribed in a circle.***



Given—P a polygon constructed with the sides a, b, c, d, and inscribed in a circle, C; and Q another polygon constructed with the same sides, but not inscriptible in a circle.

To Prove— $P > Q$.

Dem.—On the sides a, b, c, d of the polygon Q, construct circular segments equal to the segments on the corresponding sides of the polygon P.

Then the figure S has the same perimeter as the circle C.

Hence area $C >$ area S. § 360

Subtracting the circular segments from both figures, we have

$P > Q$. Q. E. D.

EXERCISES.

356. If the line AB is bisected at C, and D is a point on AC, prove that

$$\overline{AD}^2 + \overline{BD}^2 = 2\overline{AC}^2 + 2\overline{DC}^2, \text{ and}$$

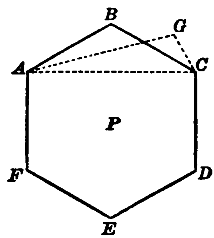
$$AD \times BD = \overline{AC}^2 - \overline{DC}^2.$$

357. Prove that the sum of the squares on the two segments of a given line is a minimum when the segments are equal.

358. Prove that the rectangle of the two segments into which a given line can be divided is a maximum when the given line is bisected.

PROPOSITION VII. THEOREM.

363. ***Of all isoperimetric polygons having the same number of sides, the maximum polygon is regular.***



Given—*ABCDEF* the maximum polygon having the given perimeter and a given number of sides.

To Prove—That *ABCDEF* is a regular polygon.

Dem.—If two of its sides, as *AG* and *GC*, were unequal, we could substitute for *AGC* the isosceles triangle *ABC*, having the same perimeter as *AGC*, but a greater area. § 357

In like manner the whole polygon could be increased without changing the length of its perimeter or the number of its sides.

For the polygon constructed with the same number of equal sides to be a maximum must be inscriptible in a circle (§ 362).

Therefore it must be a regular polygon. Q. E. D.

EXERCISES.

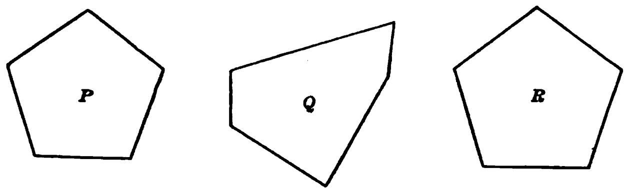
359. Find the minimum straight line between two non-intersecting circumferences.

360. Of all triangles of a given base and area the isosceles triangle has the greatest vertical angle.

361. The sum of the lines drawn from the centre of an equilateral triangle to the three vertices is less than the sum of the lines drawn from any other point to these vertices.

PROPOSITION VIII. THEOREM.

364. ***Of all polygons having the same number of sides and the same area, the regular polygon has the minimum perimeter.***



Given—P a regular and Q any irregular polygon having the same number of sides and the same area as P.

To Prove—Perimeter of $P <$ perimeter of Q.

Dem.—Draw the regular polygon R, having the same perimeter and the same number of sides as Q.

Then $Q < R$. § 363

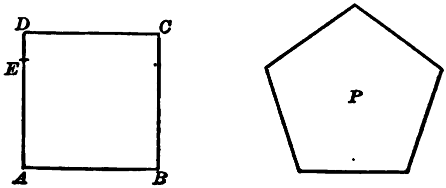
And, since the area of P equals the area of Q, $P < R$.

But of two regular polygons having the same number of sides, the one of least area has the least perimeter.

Therefore the perimeter of P is less than that of R, or less than that of Q. Q. E. D.

PROPOSITION IX. THEOREM.

365. ***Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.***



Given—$ABCD$ a square, and P an isoperimetric pentagon.

To Prove— Area $P >$ area $ABCD$.

Dem.—Let E be any point in the side AD of the square.

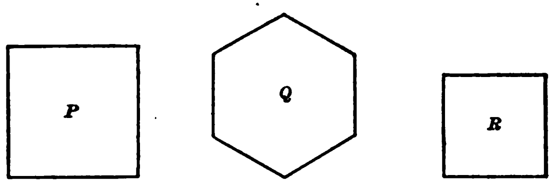
Then the square $ABCD$ may be regarded as a pentagon having the five sides AB, BC, CD, DE, and EA.

Then area $P >$ area $ABCD$. § 363

In like manner, we may prove that the area of a regular polygon of any number of sides is greater than the area of an isoperimetric polygon having one side less. Q. E. D.

Proposition X. Theorem.

366. ***If a regular polygon be constructed with a given area, its perimeter will be the less the greater the number of its sides.***



Given—P and Q regular polygons having the same area, but Q having the greater number of sides.

To Prove—Perimeter of $P >$ perimeter of Q.

Dem.—Construct R, having the same perimeter as Q and the same number of sides as P.

Then $Q > R$. § 365

And, since the area of P equals the area of Q,

$$P > R.$$

Therefore the perimeter of P is greater than that of R, or greater than that of Q. Q. E. D.

SOLID GEOMETRY.

BOOK VI.

PLANES AND POLYHEDRAL ANGLES.

367. A **Plane** is a surface such that a straight line joining any two points in it lies wholly in the surface.

368. A plane is *determined* by given lines or points when one plane, and only one, can be drawn through them.

369. A straight line is *Perpendicular to a Plane* when it is perpendicular to every straight line of the plane passing through its foot.

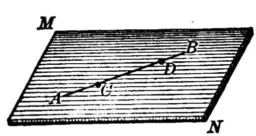
370. A straight line is *Parallel to a Plane* when it cannot meet the plane though both be indefinitely produced.

371. *Two planes are parallel* when they cannot meet though both be produced indefinitely.

LINES AND PLANES.

PROPOSITION I. THEOREM.

372. *If a straight line has two of its points in a plane, it lies wholly in that plane.*



Given—*AB* a straight line having two points, *C* and *D*, in the plane *MN*.

To Prove—That *AB* lies wholly in the plane *MN*.

Dem.—Connect the two points *C* and *D* by a straight line, *CD*.

Then *CD* lies wholly in the plane. § 367

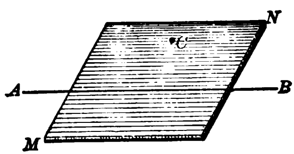
But *AB* coincides with *CD*. Ax. 6.

Hence *AB* lies wholly in the plane. Q. E. D.

Proposition II. Theorem.

373. *A plane is determined—*

1. By a straight line and a point without the line.
2. By three points not in the same straight line.
3. By two straight lines which intersect.
4. By two parallel lines.



Given—*C* a point without the straight line *AB*.

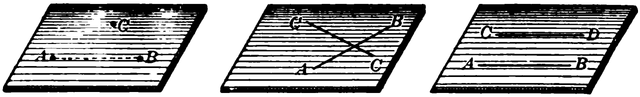
To Prove—That a plane is determined by the line *AB* and the point *C*.

Dem.—For if any plane, as *MN*, is passed through *AB*, it may be revolved about *AB* as an axis until it contains the point *C*.

Now, if the plane is turned in either direction about *AB*, it will cease to contain the point *C*.

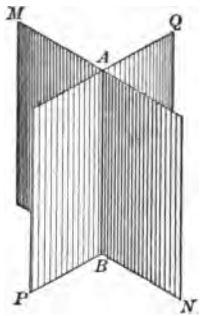
The plane is therefore determined by the given straight line and the point without it. Q. E. D.

The remaining parts of the theorem are demonstrated like the first part. Give the proof.



PROPOSITION III. THEOREM.

374. ***The intersection of two planes is a straight line.***



Given—The planes MN and PQ intersect in the line AB.

To Prove—That AB is a straight line.

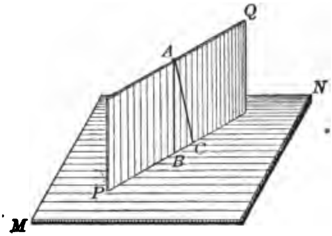
Dem.—Draw a straight line connecting the points A and B.

This line will lie in MN and in PQ. § 372

Since the straight line AB lies in both planes, it must be their intersection. Q. E. D.

PROPOSITION IV. THEOREM.

375. ***From a point without a plane only one perpendicular can be drawn to the plane.***



Given—AB a perpendicular to the plane MN from the point A.

To Prove—That only one perpendicular can be drawn to the plane MN.

Dem.—If possible, let AC be another perpendicular to the plane MN.

Since AB and AC intersect, a plane may be passed through them (§ 373), and its intersection, BC, with the plane MN is a straight line. § 374

We shall then have two perpendiculars, AB and AC, drawn from the same point to the same straight line, which is impossible. § 26

Hence only one perpendicular can be drawn from the point A to the plane MN. Q. E. D.

EXERCISES.

362. If two planes have three points in common, when will they coincide? When will they not coincide?

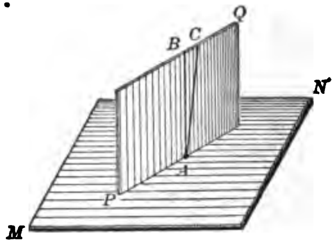
363. Can three planes contain the same straight line? Draw a figure to illustrate.

364. What is the only rectilinear polygon that is always plane? Why?

365. Hold two pencils in such a way that a plane cannot be passed through them. Hold them in two different ways so that a plane can be passed through each position.

PROPOSITION V. THEOREM.

376. *From a given point in a plane only one perpendicular can be erected to the plane.*



Given—AB a perpendicular to MN from the point A.

To Prove—That only one perpendicular can be erected to the plane MN.

Dem.—If possible, let AC be another perpendicular to the plane MN, erected at the point A.

Since AB and AC intersect, a plane may be passed through them (§ 373), and its intersection AP with the plane MN is a straight line (§ 374). We shall then have two perpendiculars, AB and AC, erected from the same point in a straight line, which is impossible. § 14

Hence only one perpendicular can be erected from the point A in the plane MN. Q. E. D.

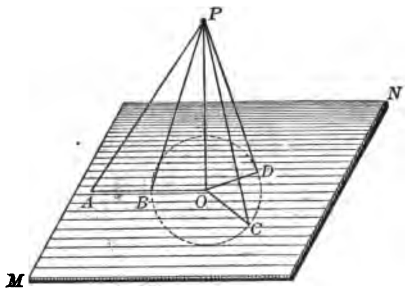
PROPOSITION VI. THEOREM.

377. *If, from a point without a plane, a perpendicular and oblique lines are drawn to the plane—*

1. The perpendicular is shorter than any one of the oblique lines.

2. Oblique lines drawn from the point, meeting the plane at equal distances from the foot of the perpendicular, are equal.

3. Of two oblique lines drawn from the point, the one meeting the plane at the greater distance from the foot of the perpendicular is the greater.



Given—P a point without the plane MN, PO a perpendicular, PA, PC, and PD oblique lines drawn to the plane.

To Prove, 1st.—PO shorter than any oblique line PC.

Dem.—Since PO and PC intersect, a plane may be passed through them (§ 373), and its intersection OC with the plane MN is a straight line. § 374

Then $PO < PC$. § 28

To Prove, 2d.—That PD and PC, which terminate at equal distances from O, are equal.

Dem.—In the right triangles POC and POD

$CO = DO$, and PO is common. Hyp.

Hence $PC = PD$. § 55

To Prove, 3d.—That if PA terminates at a greater distance from O than PC, $PA > PC$.

Dem.—On AO take $BO = CO$, and draw PB.

Then $PA > PB$. § 28

But $PB = PC$. Part 2d.

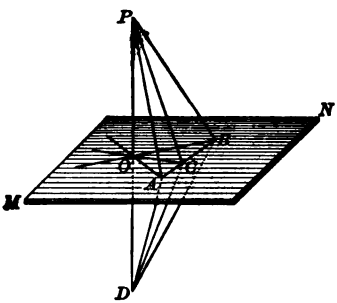
Therefore $PA > CP$. Q. E. D.

378. Cor. 1.—*Equal oblique lines drawn from a point to a plane cut off equal distances from the foot of the perpendicular.*

379. Cor. 2.—*Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.*

Proposition VII. Theorem.

380. ***If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of those lines.***



Given—PO perpendicular to AO and BO at their intersection O.

To Prove—PO perpendicular to the plane MN which contains those lines.

Dem.—Through O draw OC, any other straight line of the plane MN; and through C draw AB, cutting AO and BO in the points A and B.

Produce PO to D, making $DO = PO$, and join P and D to each of the points A, C, B.

Since AO is perpendicular to PD at its middle point,

$PA = DA$, § 22

Also $PB = DB$. § 22

Hence $\triangle APB = \triangle ADB$, § 57

And $\triangle PAC = \triangle DAC$, $\therefore PC = DC$. §§ 55, 45 (*a*).

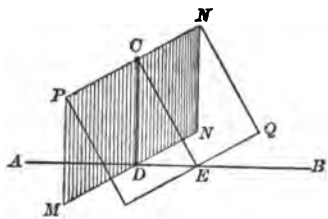
Then, since the line OC has two points, O and C, each equally distant from the extremities of PD, OC is perpendicular to PD or PO. § 25

Hence PO is perpendicular to any line OC passing through its foot in the plane MN, and is therefore perpendicular to the plane. Q. E. D.

381. COR.—CONVERSELY.—*If two straight lines are perpendicular to a given straight line at the same point, the plane of those lines is perpendicular to the given line.*

PROPOSITION VIII. THEOREM.

382. ***Through a given point without a straight line but one plane perpendicular to the line can be passed.***



Given—C a given point without the line AB.

To Prove—That but one plane can be passed through C perpendicular to AB.

Dem.—Pass the plane MN through C perpendicular to AB.

If another plane can be passed through C perpendicular to AB, let PQ be that plane.

Draw CD and CE in the planes MN and PQ respectively.

16

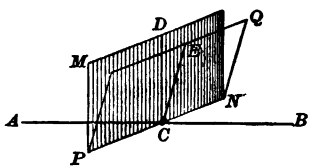
Then, since AB is perpendicular to both MN and PQ, it is perpendicular to CD and CE. § 369

We should then have two perpendiculars drawn from the same point to the same straight line, which is impossible. § 26

Hence only one plane can be passed through C perpendicular to AB. Q. E. D.

Proposition IX. Theorem.

383. *Through a given point in a straight line but one plane perpendicular to the line can be passed.*



Given—C a point in the line AB.

To Prove—That but one plane can be passed through C perpendicular to AB.

Dem.—Pass the plane MN through C perpendicular to AB.

If another plane can be passed through C perpendicular to AB, let PQ be that plane.

Draw CD and CE in the planes MN, PQ, and DCB.

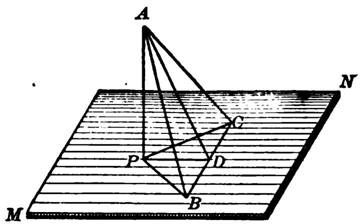
Then, since AB is perpendicular to both MN and PQ, it is perpendicular to CD and CE. § 369

We should then have two perpendiculars drawn from the same point in the same straight line, which is impossible. § 14

Hence only one plane can be passed through C perpendicular to AB. Q. E. D.

Proposition X. Theorem.

384. ***If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line of the plane, and the point of intersection is joined with any point of the perpendicular, this last line will be perpendicular to the line of the plane.***



Given—AP perpendicular to the plane MN, PD drawn perpendicular to any line BC in MN, and D, the point of intersection, joined with any point A in AP.

To Prove—AD perpendicular to BC.

Dem.—Take $DB = DC$, and draw PB, PC, AB, and AC.

Since PD is perpendicular to BC, and $BD = CD$, we have

$$PB = PC. \qquad \S\ 22$$

Then in the two triangles APB and APC, since $PB = PC$ and AP is common, we have

$$AB = AC. \qquad \S\ 55$$

Therefore the line AD has two points, A and D, each equally distant from B and C.

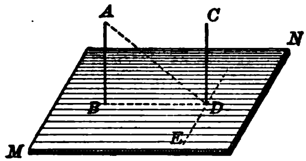
Hence AD is perpendicular to BC. § 25. Q. E. D.

385. Cor.—*The line BC is perpendicular to the plane of the triangle APD.*

For it is perpendicular to AD and PD at their point of intersection D. § 380

PROPOSITION XI. THEOREM.

386. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Given—*AB* and *CD* parallel, and *AB* perpendicular to *MN*.

To Prove—*CD* perpendicular to *MN*.

Dem.—Pass a plane through *AB* and *CD* (§ 373, 4), cutting *MN* in *BD*; draw *AD*, and in the plane *MN* draw *ED* perpendicular to *BD*.

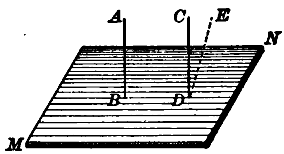
Then *AD* is perpendicular to *ED*, § 384

And *ED* is perpendicular to the plane of *ABDC*. § 385

Since *AB* and *CD* are parallel, and *ABD* is a right angle, then *CDB* is a right angle. § 36

Hence *CD* is perpendicular to *BD* and *ED* at their point of intersection, and is therefore perpendicular to *MN*. § 380. Q. E. D.

387. COR. 1.—*Two lines which are perpendicular to the same plane are parallel to each other.*



Given—*AB* and *CD* perpendicular to the plane *MN*.

To Prove—*AB* and *CD* parallel.

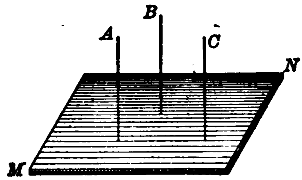
Dem.—If *AB* and *CD* are not parallel, draw *ED* parallel to *AB*.

Then *ED* will be perpendicular to *MN*. § 386

We shall then have two perpendiculars to the plane *MN* from the same point, which is impossible. § 376

Hence *AB* and *CD* are parallel. Q. E. D.

388. COR. 2.—*Two straight lines which are parallel to a third line are parallel to each other.*



Given—*A* and *B* parallel to *C*.

To Prove—*A* and *B* parallel.

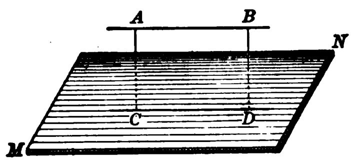
Dem.—Pass a plane *MN* perpendicular to *C*.

Then *MN* is perpendicular to *A* and *B*. § 386

Hence *A* and *B* are parallel. § 387. Q. E. D.

PROPOSITION XII. THEOREM.

389. ***If a straight line is parallel to a line of a plane, it is parallel to that plane.***



Given—*AB* parallel to the line *CD* of the plane *MN*.

To Prove—*AB* parallel to the plane *MN*.

Dem.—Pass a plane through *AB* and *CD* (§ 373, 4).

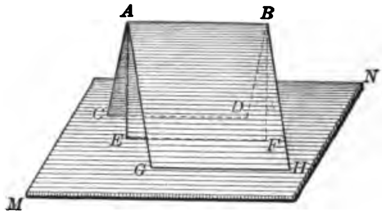
If *AB* is not parallel to *MN*, it will meet the plane *MN* in some point of *CD*, since *AB* lies in the plane *AD*.

But *AB* cannot meet *CD*, since they are parallel; hence *AB* cannot meet *MN*.

Therefore *AB* is parallel to the plane *MN*. § 370. Q. E. D.

PROPOSITION XIII. THEOREM.

390. ***If a straight line and a plane are parallel, the intersections of the plane with planes passed through the line are parallel to that line and to one another.***



Given—*AB* parallel to the plane *MN*, and let *CD*, *EF*, and *GH* be the intersections of *MN* with planes passed through *AB*.

To Prove—*AB* parallel to *CD*, *EF*, and *GH*.

Dem.—*AB* and *CD* lie in the same plane.

Since *AB* cannot meet the plane *MN*, it cannot meet *CD*, a line of the plane *MN*.

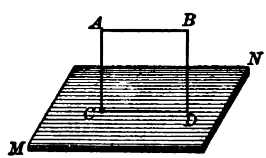
Therefore *AB* and *CD* are parallel.

In like manner, *EF*, *GH*, etc. are parallel to *AB*.

Also *CD*, *EF*, *GH*, etc. are parallel to each other.

§ 388. Q. E. D.

391. COR.—*If a straight line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*



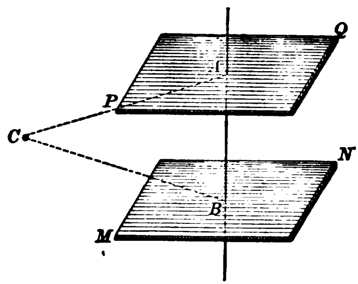
Given—*AB* parallel to the plane *MN*, and *CD* a line drawn through any point *C* of the plane *MN*.

To Prove—That *CD* lies in the plane *MN*.

Dem.—Pass a plane through AB and the point C (§ 373, 1), intersecting the plane in a parallel to AB (§ 390), which must coincide with CD, since only one line can be drawn through the point C parallel to AB. Ax. 7. Q. E. D.

PROPOSITION XIV. THEOREM.

392. *Two planes perpendicular to the same straight line are parallel.*



Given—The planes MN and PQ perpendicular to the line AB.

To Prove—MN and PQ parallel.

Dem.—If MN and PQ are not parallel, they will meet if sufficiently produced.

Let C be a point in their intersection, and we shall then have two planes drawn through C perpendicular to AB, which is impossible. § 382.

Therefore MN and PQ are parallel. § 371. Q. E. D.

EXERCISES.

366. How many planes are determined by four points in space, no three of which are in the same straight line?

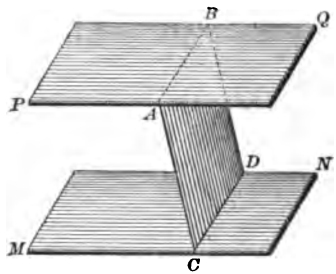
367. Why is a three-legged stool always stable on the floor while a four-legged table may not be?

368. If one of two parallel lines is parallel to a plane, what is true of the other? Prove it.

369. State four conditions that determine a plane.

PROPOSITION XV. THEOREM.

393. ***If two parallel planes are cut by a third plane, the intersections are parallel.***



Given—The parallel planes *MN* and *PQ* cut by the plane *AD* in the lines *CD* and *AB* respectively.

To Prove—*AB* and *CD* parallel.

Dem.—*AB* and *CD* lie in the plane *AD*.

And, since the planes *MN* and *PQ* cannot meet, however far they are produced, *AB* and *CD* cannot meet if produced indefinitely.

Therefore *AB* and *CD* are parallel. § 31. Q. E. D.

394. COR.—*Parallel lines included between parallel planes are equal.*

Given—*AC* and *BD* parallel lines included between the parallel planes *MN* and *PQ*.

To Prove— $AC = BD$.

Dem.—Pass a plane through *AC* and *BD* (§ 373, 4) intersecting *MN* and *PQ* in the parallel lines *CD* and *AB* respectively (§ 393).

Then *ABDC* is a parallelogram, § 74

And $AC = BD$. § 81.

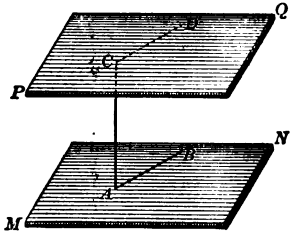
EXERCISE.

370. If a straight line intersect two parallel planes, it makes equal angles with them.

371. If a plane bisects a straight line at right angles, any point in the plane is equally distant from the extremities of the line.

PROPOSITION XVI. THEOREM.

395. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*



Given—AC perpendicular to the plane MN, which is parallel to the plane PQ.

To Prove—AC perpendicular to PQ.

Dem.—Through A draw any straight line AB in the plane MN.

Through AB and AC pass a plane (§ 373, 3), intersecting MN and PQ in the parallel lines AB and CD respectively (§ 393).

Then AC is perpendicular to CD. § 36

But CD is any line of the plane PQ passing through the foot of AC.

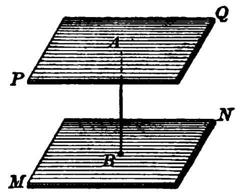
Therefore AC is perpendicular to PQ. § 369. Q. E. D.

396. COR. 1.—*Two parallel planes are everywhere equally distant.*

For all perpendiculars to the planes are parallel. § 387

Therefore they are equal. § 394

397. Cor. 2.—*Through a given point a plane can be drawn parallel to a given plane, and but one.*



Given—A any point, and MN any plane.

To Prove—That a plane can be drawn through A parallel to MN.

Dem.—Draw AB perpendicular to MN.

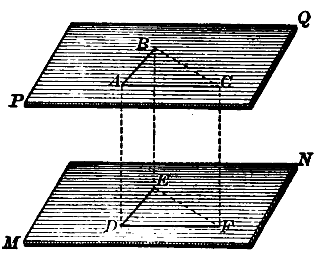
Then through A one plane, and but one, can be passed perpendicular to AB (§ 383); let PQ be that plane.

Then PQ and MN are parallel. § 392

Therefore but one plane can be drawn through A parallel to MN. Q. E. D.

Proposition XVII. Theorem.

398. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Given—PQ and MN the planes of the angles BAC and EDF, and AB and AC parallel to DE and DF respectively.

To Prove, 1st—$\angle BAC = \angle EDF$.

Dem.—Take $AB = DE$ and $AC = DF$, and draw BC, EF, AD, BE, and CF.

Since AC and DF are equal and parallel, the figure $ADFC$ is a parallelogram. § 83

Hence AD is equal and parallel to CF. § 79

In like manner, AD is equal and parallel to BE.

Then BE is equal and parallel to CF. Ax. 1. § 33

Hence $BEFC$ is a parallelogram, and $BC = EF$. § 79

Therefore $\triangle ABC = \triangle DEF$, § 57

And $\angle BAC = \angle EDF$. § 45 (a)

To Prove, 2d—PQ parallel to MN.

Dem.—Since two lines which intersect determine the position of a plane, § 373, 3

Then the position of PQ is determined by AB and AC, and that of MN is determined by DE and DF.

Since AB and AC are parallel to DE and DF, respectively,

Then the planes MN and PQ are parallel. Q. E. D.

EXERCISES.

372. If two parallel lines intersect a plane, they make equal angles with it.

373. Find the locus of all points at a given distance from a given plane.

374. From a given point without a plane construct a perpendicular to the plane.

375. From a given point in a plane erect a perpendicular to the plane.

376. Find the locus of all points in space equally distant from any two given points.

377. Find the locus—

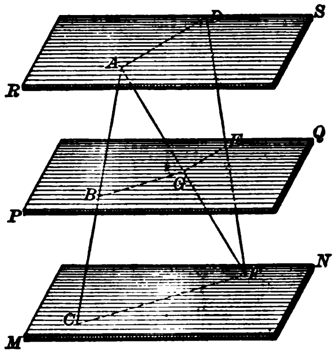
a. Of all points equally distant from two parallel planes.

b. Of all points in space equally distant from the extremities of a given straight line.

c. Of all points in space equally distant from three given points not in the same straight line.

PROPOSITION XVIII. THEOREM.

399. *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*



Given—AC and DF two lines intersected by the parallel planes MN, PQ, and RS in the points A, B, C, and D, E, F.

To Prove— $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

Dem.—Draw AF, and pass a plane through AC and AF (§ 373, 3), intersecting the planes PQ and MN in BG and CF.

Then BG and CF are parallel. § 393

In like manner, GE is parallel to AD.

Hence $\dfrac{AB}{BC} = \left(\dfrac{AG}{GF}\right) = \dfrac{DE}{EF}$, § 242

And $\dfrac{AB}{BC} = \dfrac{DE}{EF}$. Ax. 1. Q. E. D.

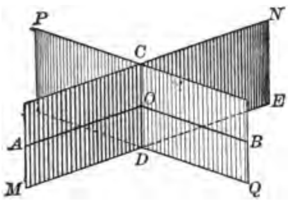
400. COR. 1.—*If two lines are cut by any number of parallel planes, the corresponding segments are proportional.*

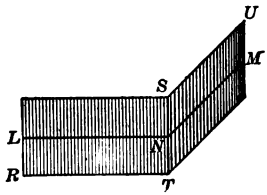
401. COR. 2.—*If any number of lines are cut by three parallel planes, their corresponding segments are proportional.*

DIHEDRAL ANGLES.

402. **A Dihedral Angle** is the angle formed by two planes which intersect each other.

Thus, the planes *MN* and *PQ* meeting in *CD* form a





dihedral angle; also *RS* and *TU* meeting in *ST* form a dihedral angle.

The planes *RS* and *TU* are called the *faces*, and *TS* the *edge*, of the dihedral angle.

A dihedral angle may be designated by the two letters on its edge; or if two or more dihedral angles have a common edge, by four letters, one in each face and two in the edge.

Thus, we have the dihedral angle *ST* or the dihedral angle *MDCQ*.

403. The *Plane Angle* of a dihedral angle is the angle formed by drawing a line in each plane perpendicular to the line of intersection at the same point.

Thus, if *AO* and *BO* are in the faces *MC* and *QC* respectively, perpendicular to *CD* at *O*, then *AOB* is the plane angle of the dihedral angle *MDCQ*.

Hence it is evident—

1st. *That the plane angle is the same at whatever point of the edge of the dihedral angle it is formed* (§§ 35, 398).

2d. *That the plane of the plane angle AOB is perpendicular to the edge CD* (§ 373, 3).

404. Two dihedral angles are *equal* when they can be placed so that their faces coincide. Hence it is evident—

1st. *That two dihedral angles are equal when their plane angles are equal.*

2d. *That the plane angles of equal dihedral angles are equal.*

405. Two dihedral angles are *adjacent* when they have a common edge and a common plane between them.

Thus, the dihedral angles *MDCQ* and *QDCE* (§ 402) are adjacent.

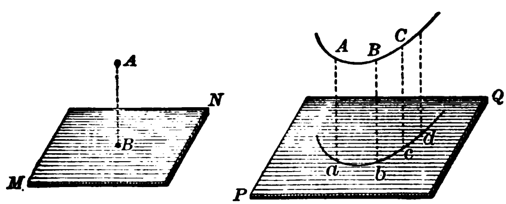
406. Two dihedral angles are *vertical* when the faces of one are the extension of the faces of the other.

Thus, the dihedral angles *PCDN* and *MDCQ* (§ 402) are vertical angles.

407. When one plane meets another, making the adjacent angles equal, the angles thus formed are *right dihedral angles*, and the planes are *perpendicular* to each other.

Thus, if the plane *QC* (§ 402) meets the plane *MN*, making the adjacent angles *MDCQ* and *QDCE* equal, they are right angles, and the planes are perpendicular.

408. The *Projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

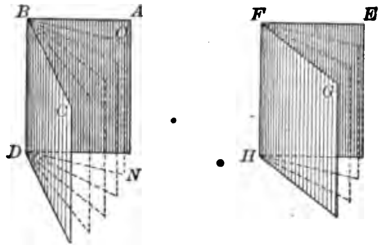


The projection of the point *A* on the plane *MN* is *B*.

The projection of the line *ABC*, etc. on the plane *PQ* is the line *abc*, etc. formed by the projection of all the points of *ABC*, etc. on the plane.

PROPOSITION XIX. THEOREM.

409. *Two dihedral angles are in the same ratio as their plane angles.*



Given—The two dihedral angles $ABDC$ and $EFHG$, and ABC and EFG their plane angles.

To Prove— $$\frac{\angle ABDC}{\angle EFHG} = \frac{\angle ABC}{\angle EFG}.$$

1st. *If the angles are commensurable.*

Dem.—Since the plane angles are commensurable, take the common unit of measure ABO, which is contained 5 times in ABC and 3 times in EFG.

Then $$\frac{\angle ABC}{\angle EFG} = \frac{5}{3}. \qquad (1)$$

Pass planes through the edges of the dihedral angles and the lines of division, thus dividing the dihedral angles into smaller dihedral angles, which are all equal to each other, since their plane angles are equal. § 404

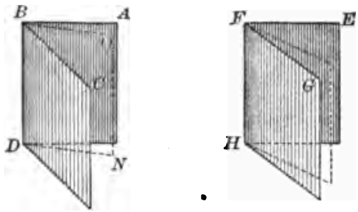
The dihedral angle $ABDC$ contains 5 of these equal angles, and $EFHG$ contains 3 of them.

Then $$\frac{\angle ABDC}{\angle EFHG} = \frac{5}{3}. \qquad (2)$$

From (1) and (2), we have

$$\frac{\angle ABDC}{\angle EFHG} = \frac{\angle ABC}{\angle EFG}. \qquad \text{Ax. 1.}$$

2d. *If the angles are incommensurable.*



Dem.—Take a unit of measure that is contained an exact number of times in $\angle EFG$. If we apply this unit to $\angle ABC$, there will be a remainder, $\angle ABO$, less than the unit of measure.

Pass a plane through BD and BO. § 373, 3

Then, since the angles EFG and CBO are commensurable, we have

$$\frac{\angle CBDO}{\angle GFHE} = \frac{\angle CBO}{\angle EFG}. \qquad \text{1st part.}$$

Now, if the unit of measure is indefinitely diminished, the remainder $\angle ABO$ will be indefinitely diminished, and $\angle CBO$ will approach $\angle CBA$ as its limit, and $\angle CBDO$ will approach $\angle CBDA$ as its limit.

Hence $$\text{lim.}\frac{\angle CBDO}{\angle GFHE} = \text{lim.}\frac{\angle CBO}{\angle EFG}. \qquad \text{§ 140}$$

But $$\text{lim.}\frac{\angle CBDO}{\angle GFHE} = \frac{\angle CBDA}{\angle GFHE},$$

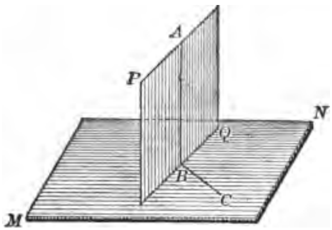
And $$\text{lim.}\frac{\angle CBO}{\angle EFG} = \frac{\angle ABC}{\angle EFG}.$$

Then $$\frac{\angle CBDA}{\angle GFHE} = \frac{\angle ABC}{\angle EFG}. \qquad \text{Ax. 1.} \quad \text{Q. E. D.}$$

410. Scholium.—Since dihedral angles have the same ratio as their plane angles, the plane angle is taken as the *measure* of the dihedral angle. See § 177.

PROPOSITION XX. THEOREM.

411. ***If a straight line is perpendicular to a plane, every plane passed through the line is also perpendicular to the first plane.***



Given—AB a line perpendicular to the plane MN.

To Prove—That any plane, as PQ, passed through AB is also perpendicular to MN.

Dem.—Draw BC, in the plane MN, perpendicular to the line of intersection BQ.

Since AB is perpendicular to the plane MN, it is perpendicular to BC and BQ. § 369

Hence $\angle ABC$ is the plane angle of the dihedral angle formed by the planes MN and PQ, and is a right angle.

Therefore the planes are perpendicular. Q. E. D.

EXERCISES.

378. If two lines are at right angles, are their projections on any plane also at right angles?

379. Through a given point only one plane can be passed parallel to a given plane.

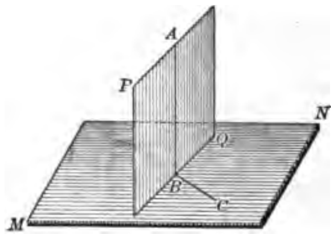
380. Prove that a line and its projection on a plane determine a second plane perpendicular to the first.

381. Two planes which are parallel to the same straight line are either parallel to each other or their intersection is parallel to this line.

382. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

PROPOSITION XXI. THEOREM.

412. *If two planes are perpendicular to each other, a straight line drawn in one of them, perpendicular to their intersection, is perpendicular to the other.*



Given—The plane *PQ* perpendicular to *MN*, and *AB* perpendicular to their intersection *BQ*.

To Prove—*AB* perpendicular to *MN*.

Dem.—Draw *BC* in *MN* perpendicular to *BQ*.

Since the plane *PQ* is perpendicular to *MN*, the plane angle *ABC* is a right angle.

Therefore *AB*, being perpendicular to *BQ* and *BC* at *B*, is perpendicular to the plane *MN*. § 380. Q. E. D.

413. COR. 1.—*If two planes are perpendicular to each other, a perpendicular to one of the planes from any point of their intersection will lie in the other plane.*

Given—The plane *PQ* perpendicular to *MN*; at *B* in their intersection draw *AB* perpendicular to *MN*.

To Prove—That *AB* lies in *PQ*.

Dem.—A line drawn in *PQ* perpendicular to *BQ* at *B* will be perpendicular to *MN*.

Then, if *AB* does not lie in the plane *PQ*, we shall have two perpendiculars to the same plane at the same point, which is impossible (§ 376). Hence *AB* lies in *PQ*.

414. COR. 2.—*If two planes are perpendicular to each other, a perpendicular to one of the planes from any point of the other will lie in the other plane.*

Given—The plane PQ perpendicular to MN, and from any point, as A of PQ, a perpendicular to MN is drawn.

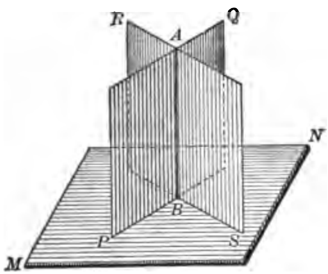
To Prove—That AB lies in PQ.

Dem.—A line drawn from A in PQ perpendicular to BQ will be perpendicular to MN.

Then, if AB does not lie in the plane PQ, we shall have two perpendiculars to the same plane from the same point, which is impossible. Hence AB lies in PQ. § 375

PROPOSITION XXII. THEOREM.

415. ***If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.***



Given—The planes PQ and RS, intersecting in the line AB, perpendicular to MN.

To Prove—That AB is perpendicular to MN.

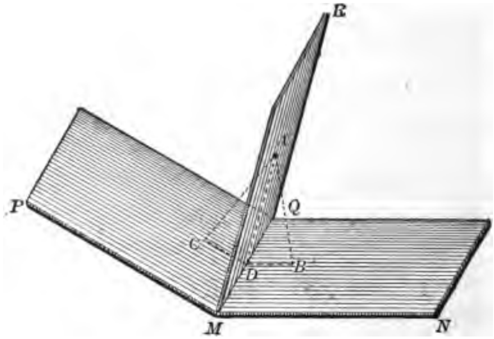
Dem.—For a perpendicular to MN at B will lie in both PQ and RS. § 413

It is therefore their line of intersection.

Hence AB is perpendicular to MN. Q. E. D.

PROPOSITION XXIII. THEOREM.

416. ***Every point in the plane which bisects a dihedral angle is equally distant from the faces of the angles.***



Given—A any point in the plane MR which bisects the dihedral angle $PMQN$.

To Prove—A equally distant from the planes PQ and NQ.

Dem.—Draw AB and AC perpendicular to the planes NQ and PQ respectively.

Pass a plane through AB and AC (§ 373, 3), making the intersections BD, AD, and CD.

The plane $ACDB$ is perpendicular to the planes NQ and PQ. § 411

Then $ACDB$ is perpendicular to MQ. § 415

Hence ADB and ADC are the plane angles of the dihedral angles $PMQR$ and $NMQR$. § 403

Whence $\angle ADB = \angle ADC$. § 404

Hence rt. $\triangle ABD =$ rt. $\triangle ACD$. § 59

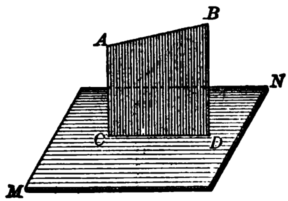
Therefore $AB = AC$, § 45 (*a*)

And A is equally distant from the planes PQ and NQ.

Q. E. D.

PROPOSITION XXIV. THEOREM.

417. ***Through any given straight line a plane can be passed perpendicular to any given plane.***



Given—AB any straight line, and MN any plane.

To Prove—That a plane can be passed through AB perpendicular to MN.

Draw AC perpendicular to MN, and pass a plane through AB and AC (§ 373, 3).

Then the plane AD is perpendicular to the plane MN. § 411

Since from A only one perpendicular can be let fall to the plane MN, and AB and AC determine the position of but one plane (§ 373, 3), then AD is the only plane that can be passed through AB perpendicular to MN, unless AB is perpendicular to MN, in which case an indefinite number of planes can be passed through AB perpendicular to MN (§ 411). Q. E. D.

EXERCISES.

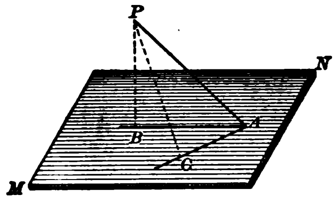
383. Any point not in the bisector of a dihedral angle is unequally distant from the faces.

384. Two lines intersected by three parallel planes are 14 and 21 inches long. If the segments of the first are 8 and 6, what are the segments of the second?

385. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars from the extremities of the other diagonal to the plane are equal.

PROPOSITION XXV. THEOREM.

418. ***The angle which a straight line makes with its own projection upon a plane is the least angle which it makes with any line of that plane.***



Given—AB the projection of AP on the plane MN, and AC any other line drawn through A in MN.

To Prove— $\angle PAB < \angle PAC$.

Dem.—Take $AC = AB$, and draw PB and PC.

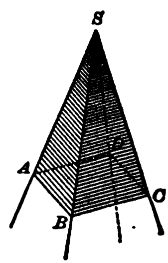
Then, in the triangles PAB and PAC, PA is common,

And $PB < PC$. § 377, 1

Hence $PAB < PAC$. § 62. Q. E. D.

POLYHEDRAL ANGLES.

419. A Polyhedral Angle is the opening between three or more planes which meet at a common point.



Thus, the angle $S—ABCD$, formed by the planes ASB, BSC, CSD, and DSA, meeting in the common point S, is a polyhedral angle.

S is the *vertex*.

SA, SB, SC, and SD are the *edges*.

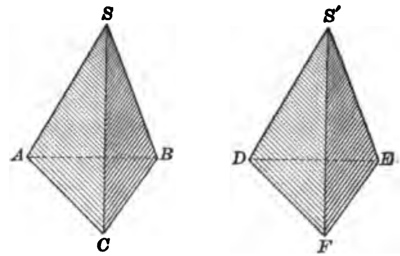
The triangles SAB, SAD, etc. are the *faces*.

The angles ASB, BSC, etc. are the *face angles*.

420. A *Trihedral Angle* is a polyhedral angle having three faces; a *Tetrahedral Angle* is a polyhedral angle having four faces, etc.

421. Two polyhedral angles are *equal* when, if applied to each other, they coincide in all their parts.

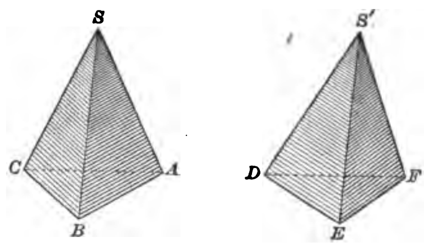
Thus, if the face angles ASB, ASC, and CSB are equal, respectively, to $DS'E$, $DS'F$, and $FS'E$, and the dihedral angles AS, BS, and CS to the dihedral angles DS', ES', and FS', respectively,



then the trihedral angles $S-ABC$ and $S'-DEF$ can be applied to each other so that they will coincide throughout, and are therefore equal.

422. Two polyhedral angles are *symmetrical* when the face angles and dihedral angles of one are equal respectively to the corresponding face angles and dihedral angles of the other, but *arranged in an inverse order.*

Thus, if the face angles $CSB = FS'E$, $BSA = ES'D$, and $ASC = DS'F$, and the dihedral angles $CS = FS'$, $BS = ES'$, and $AS = DS'$, the trihedral angles

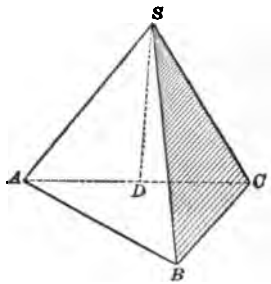


$S-ABC$ and $S'-DEF$ are symmetrical.

It is evident that two symmetrical polyhedral angles cannot be made to coincide by superposition, but they are said to be equal by symmetry.

PROPOSITION XXVI. THEOREM.

428. ***The sum of any two face angles of a trihedral angle is greater than the third.***



The theorem requires proof only when the angle considered is greater than each of the others.

Given—The trihedral angle $S-ABC$, with the face angle ASC greater than either ASB or BSC.

To Prove—$(\angle ASB + \angle BSC) > \angle ASC$.

Dem.—In the face ASC draw SD equal to SB, making the angle CSD equal to the angle CSB; draw AC through D, and then draw AB and BC.

In the triangles CSD and CSB, CS is common, and by construction $SD = SB$ and $\angle CSD = \angle CSB$.

Hence $\triangle CSD = \triangle CSB$, § 55

And $CD = CB$. § 45 (*a*)

In $\triangle ACB$, $(AB + BC) > (AD + DC)$. Ax. 5.

Subtracting the equals BC and DC,

Then $AB > AD$.

In the triangles ASB and ASD, AS is common, $SD = SB$, and $AB > AD$.

Hence $\angle ASB > \angle ASD$. § 62

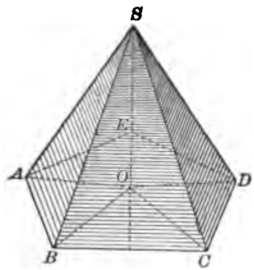
Adding BSC to the first member of this inequality, and DSC to the second member, we have

$$(\angle ASB + \angle BSC) > (\angle ASD + \angle DSC),$$

Or $$(\angle ASB + \angle BSC) > \angle ASC. \quad \text{Q. E. D.}$$

PROPOSITION XXVII. THEOREM.

424. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given—$S—ABCDE$ a convex polyhedral angle.

To Prove—$\angle ASB + \angle BSC + \angle CSD + \angle DSE + \angle ESA$ $<$ four right angles.

Dem.—Pass a plane cutting the faces of the polyhedral angle in the lines AB, BC, CD, DE, and EA.

Let O be any point in the polygon $ABCDE$, and draw AO, BO, CO, DO, and EO.

Then $\angle SBA + \angle SBC > \angle ABO + \angle CBO.$ § 423

For the same reason,

$$\angle SCB + \angle SCD > \angle BCO + \angle DCO, \text{ etc.}$$

Adding these inequalities, we have the sum of the angles at the bases of the triangles whose vertex is S greater than the sum of the angles at the bases of the triangles whose vertex is O.

But the sum of *all* the angles of the triangles whose vertex is S is equal to the sum of *all* the angles of the triangles whose vertex is O. § 48

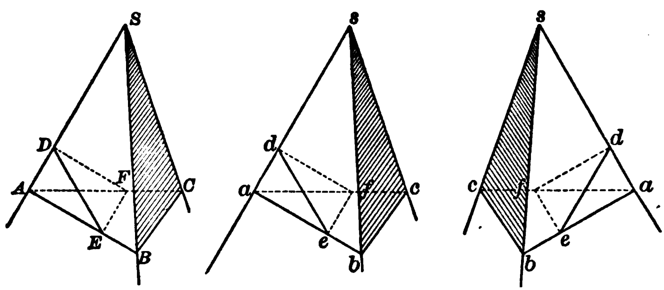
Subtracting the angles at the bases from these equal sums, the sum of the angles at S is less than the sum of the angles at O.

But the sum of the angles at O equals four right angles. § 19

Hence the sum of the angles at S is less than four right angles. Q. E. D.

PROPOSITION XXVIII. THEOREM.

425. ***If two trihedral angles have the three face angles of the one respectively equal to the three face angles of the other, the corresponding dihedral angles are equal.***



Given—$\angle ASB = \angle asb$, $\angle BSC = \angle bsc$, and $\angle ASC = \angle asc$.

To Prove—The dihedral angle SA = the dihedral angle as.

Dem.—Lay off the six equal distances SA, SB, SC, sa, sb, and sc, and draw AB, BC, CA, ab, bc, and ca.

Then $\triangle ASB = \triangle asb$, § 55

And $AB = ab$. § 45 (*a*)

In like manner, $BC = bc$, and $AC = ac$.

Then $\triangle ABC = \triangle abc$, § 57

And $\angle A = \angle a$. § 45 (*a*)

Take $AD = ad$, and draw DE in the face ASB and DF in the face ASC, perpendicular to SA; these lines meet AB and AC respectively, since the triangles SAB and SAC are isosceles. Draw EF. On sa take $ad = AD$, and construct def as DEF was constructed.

In the triangle ADE and ade, $AD = ad$ and $\angle A = \angle a$.

Hence $\triangle ADE = \triangle ade$. § 60

Then $AE = ae$, and $DE = de$.

In like manner, $AF = af$, and $DF = df$.

Whence $\triangle AEF = \triangle aef$, § 55

And $EF = ef$.

Finally, $\triangle DEF = \triangle def$, § 57

And $\angle EDF = \angle edf$.

Therefore the dihedral angle AS equals the dihedral angle as. § 404, 1

In a similar manner, the dihedral angles SB and SC can be proved equal to the dihedral angles sb and sc respectively. Q. E. D.

426. SCHOLIUM.—If the face angles of the trihedral angles are similarly placed, the trihedral angles may be applied to each other and will coincide in all their parts, and will therefore be equal (§ 421).

If the face angles are not similarly placed, the trihedral angles will not coincide, but will be symmetrical (§ 422).

Both cases are represented in the figure.

427. COR.—*If the symmetrical trihedral angles are isosceles, they are always equal.*

EXERCISES.

ORIGINAL THEOREMS.

386. If a plane meets another plane, the sum of the adjacent dihedral angles is equal to two right angles.

387. If two planes intersect each other, the opposite or vertical dihedral angles are equal to each other.

388. If a plane intersects two parallel planes,

a. The alternate-interior dihedral angles are equal.

b. The exterior-interior dihedral angles are equal.

c. The sum of the interior dihedral angles on the same side is equal to two right angles.

389. If two dihedral angles have their faces parallel, they are either equal or supplementary.

390. If a plane is perpendicular to a line at its middle point, any point in the plane is equally distant from the extremities of the line, and any point out of the plane is unequally distant from the extremities of the line.

391. If two planes are parallel to a third plane, they are parallel to each other.

392. If a line is parallel to one plane and perpendicular to another, these two planes are perpendicular.

393. If two planes which intersect contain two lines parallel to each other, the intersection of the planes is parallel to the lines.

394. The three planes which bisect the dihedral angles of a trihedral angle meet in the same straight line.

395. The three planes passed through the bisectors of the three face angles of a trihedral angle, and perpendicular to these faces respectively, intersect in the same straight line.

396. The three planes passed through the edges of a trihedral angle, perpendicular to the opposite face angles, intersect in the same straight line.

397. The three planes passed through the edges of a trihedral angle and the bisectors of the opposite face angles intersect in the same straight line.

398. Suppose a polyhedral angle formed by three, four, five equilateral triangles. What is the sum of the face angles at the vertex?

399. If the edges of one polyhedral angle are respectively perpendicular to the faces of a second polyhedral angle, then the edges of the latter are respectively perpendicular to the faces of the former.

BOOK VII.

POLYHEDRONS.

428. A **Polyhedron** is a solid bounded by planes.

The planes which bound the polyhedron are the *faces.*

The intersections of the faces are the *edges.*

The intersections of the edges are the *vertices.*

A straight line joining any two vertices not in the same face is a *diagonal.*

429. A *Tetrahedron* is a polyhedron of four faces.

A *Hexahedron* is a polyhedron of six faces.

An *Octahedron* is a polyhedron of eight faces.

A *Dodecahedron* is a polyhedron of twelve faces.

An *Icosahedron* is a polyhedron of twenty faces.

430. A polyhedron is *convex* when the section formed by any plane intersecting it is a convex polygon.

The polyhedrons treated of in this work are convex unless otherwise stated.

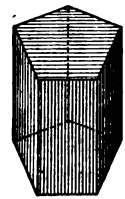
431. The **volume** of any polyhedron is the numerical measure of its magnitude referred to some other polyhedron regarded as the *unit.*

The *unit of volume* for measuring polyhedrons is generally a *cube* whose edges are some linear unit; the volume of a polyhedron is therefore expressed by the number of cubic units it contains.

432. Equivalent Solids are those which have equal volumes. They may differ in form.

THE PRISM.

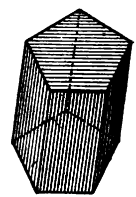
433. A **Prism** is a polyhedron two of whose faces are equal and parallel polygons, and whose faces are parallelograms.



The parallel faces are the *bases;* the remaining faces are the *lateral area;* the intersections of the faces are the *lateral edges;* the perpendicular distance between the bases is the *altitude.*

434. Prisms are named from their bases. A *Triangular Prism* is one whose bases are triangles. A *Quadrangular Prism* is one whose bases are quadrilaterals, etc.

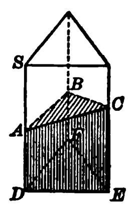
435. A *Right Prism* is one whose lateral edges are perpendicular to the bases.



An *Oblique Prism* is one whose lateral edges are not perpendicular to the bases.

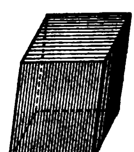
A *Regular Prism* is a right prism whose bases are regular polygons. § 312

436. If the prism $S-DEF$ is cut by a plane not parallel to the base, that portion of the prism included between the base and the plane, as $ABC-DEF$, is a *Truncated Prism.*

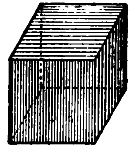


437. A *Right Section* of a prism is the section made by passing a plane perpendicular to the lateral edges.

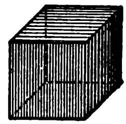
438. A **Parallelopiped** is a prism whose bases are parallelograms. All the faces are therefore parallelograms.



439. A *Right Parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.



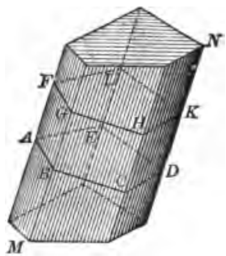
A *Rectangular Parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.



440. A **Cube** is a rectangular parallelopiped whose edges and faces are all equal.

PROPOSITION I. THEOREM.

441. *The sections of a prism made by parallel planes are equal polygons.*



Given—*AD* and *FK* parallel planes cutting the prism *MN*.

To Prove—*AD* and *FK* equal polygons.

Dem.—Since *AD* and *FK* are parallel planes, *AB* is parallel to *FG*, *BC* to *GH*, etc. § 393

Hence $AB = FG$, $BC = GH$, etc., § 81

$\angle ABC = \angle FGH$, $\angle BCD = \angle GHK$, etc. § 398

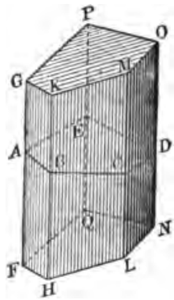
Hence the polygons *AD* and *FK* are both equilateral and equiangular.

Therefore *AD* and *FK* are equal polygons. § 215. Q. E. D.

442. COR.—*Any section of a prism made by a plane parallel to the base is equal to the base.*

PROPOSITION II. THEOREM.

443. *The lateral area of a prism is equal to the product of the perimeter of a right section of the prism by a lateral edge.*



Given—AD a right section of the prism HP.

To Prove—The lateral area $= HK \times (AB + BC + CD + DE + AE)$.

Dem.—Since AD is a right section (§ 437), AB is perpendicular to FG.

Hence the area $FHKG = HK \times AB$. § 229

In like manner, area $HLMK = HK \times BC$,

area $LNOM = HK \times CD$, etc.

Adding, the lateral area $= HK \times (AB + BC + CD + DE + AE)$. Q. E. D.

444. COR.—*The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

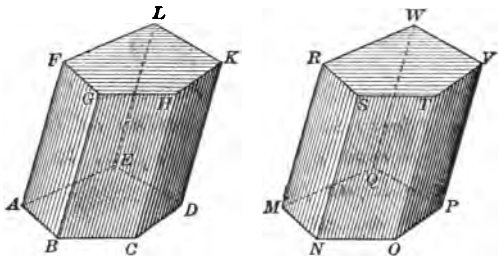
EXERCISES.

400. Find the lateral area of a right pentangular prism, each of the sides of the base being 10 in. and the altitude 12 in.

401. Find the entire surface of a right triangular prism the sides of whose base are 8, 12, and 16 in., respectively, and altitude 20 in.

PROPOSITION III. THEOREM.

445. ***Two prisms are equal if three faces including a trihedral angle of the one are respectively equal to three faces similarly placed including a trihedral angle of the other.***



Given—The trihedral angles H and T, formed by the equal faces $HL = TW$, $HB = TN$, $HD = TP$.

To Prove—The prisms equal.

Dem.—Since $\angle GHK = \angle STV$, $\angle GHC = \angle STO$, and $\angle KHC = \angle VTO$, the trihedral angle $H-GKC =$ the trihedral angle $T-SVO$. § 426

Apply the prism MV to the prism AK, so that the face TW coincides with its equal HL.

Then dihedral angle $ST =$ dihedral angle GH, § 425

Also dihedral angle $TV =$ dihedral angle HK. § 425

Hence the face TN coincides with the face HB, and TP with HD.

Then the edge ON coincides with the edge CB, and OP with CD.

Hence the planes of BCD and NOP will coincide. § 398

Now, since the bases are equal, they will coincide throughout.

Therefore the prisms are equal. Q. E. D.

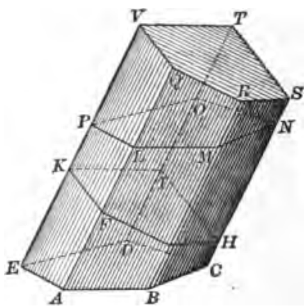
446. COR. 1.—*Two right prisms are equal when they have equal bases and equal altitudes.*

For the equal faces may always be similarly placed.

447. SCHOLIUM.—The demonstration above applies to two *truncated prisms*, without change.

PROPOSITION IV. THEOREM.

448. ***Any oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.***



Given—FI a right section of the oblique prism AO.

To Prove—The prism AO equal to the right prism FT, whose altitude FQ equals the lateral edge AL.

Dem.—Produce AL to Q, making $FQ = AL$.

Pass a plane through Q parallel to FI, meeting the edges produced in R, S, T, and V, completing the right prism FT.

It is evident that

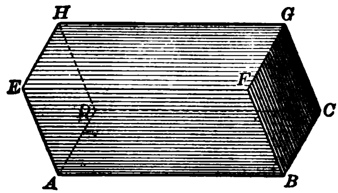
The truncated prism $AI =$ the truncated prism LT. § 447

Adding the solid FO to both members of this equation,

The oblique prism $AO =$ the right prism FT. Q. E. D.

PROPOSITION V. THEOREM.

449. *The opposite faces of a parallelopiped are equal and parallel.*



Given—AG a parallelopiped.

To Prove—The faces AF and DG equal and parallel.

Dem.—AB is equal and parallel to DC. § 79

In like manner, AE is equal and parallel to DH.

Hence $\angle EAB = \angle HDC$,

and the faces AF and DG are parallel. § 398

Therefore the face $AF =$ the face DG. § 90. Q. E. D.

In a similar manner we may prove AH equal and parallel to BG.

450. COR.—*Either face of a parallelopiped may be taken as the base.*

EXERCISES.

402. What is the locus of all points equidistant from two consecutive faces of a polyhedron?

403. In how many ways can a polyhedral angle be formed with equiangular triangles? With squares?

404. In how many ways can a polyhedral angle be formed with equiangular pentagons? With equiangular hexagons?

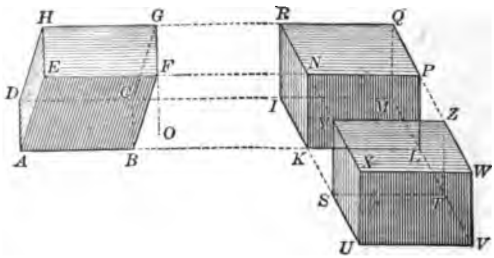
405. Can a polyhedral angle be formed by equiangular heptagons?

406. Find the entire surface of a regular hexagonal prism, each side of the base being 6 in. and the altitude 8 in.

407. Find the entire surface of a regular triangular prism, each side of the base being 4 ft. and the altitude 3 chains.

Proposition VI. Theorem.

451. ***Any parallelopiped is equivalent to a rectangular parallelopiped of the same altitude and equivalent base.***



Given—AG an oblique parallelopiped whose base is $ABCD$ and altitude FO.

To Prove—AG equivalent to the right parallelopiped UZ, having the same altitude and an equivalent base.

Dem.—Produce the edges AB, DC, EF, and HG.

Take $NP = EF$, and pass the planes NI and PM perpendicular to NP, forming the right parallelopiped KQ.

Then parallelopiped $AG \Leftrightarrow$ parallelopiped KQ. § 448

Produce the edges IK, ML, RN, and QP.

Take $YX = RN$, and pass the planes YT and XV perpendicular to YX, forming the right parallelopiped UZ.

Then parallelopiped $KQ \Leftrightarrow$ parallelopiped UZ. § 448

But UZ is a rectangular parallelopiped.

Therefore AG is equivalent to the rectangular parallelopiped UZ.

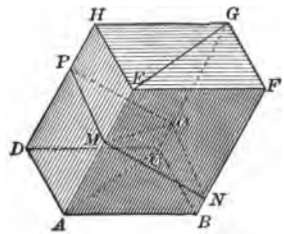
Now, base $AC \Leftrightarrow$ base $KM \Leftrightarrow UT$, § 230, Cor. 4

Also $FO = YS$. §§ 387, 394

Hence AG and UZ have the same altitude and equivalent bases. Q. E. D.

PROPOSITION VII. THEOREM.

452. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given—$ACGE$ a plane passed through the diagonally opposite edges of the parallelopiped AG.

To Prove—Prism $ABC-F \Leftrightarrow$ prism $ACD-H$.

Dem.—Let $MNOP$ be a right section of the parallelopiped AG, cutting the plane $ACGE$ in MO.

The diagonal MO divides the parallelogram $MNOP$ into two equal triangles. § 80

The oblique prism $ABC-F$ is equivalent to a right prism whose base is the triangle MNO, and whose altitude is AE. § 448

In like manner, the oblique prism $ADC-H$ is equivalent to a right prism whose base is MOP, and whose altitude is AE.

But the two right prisms are equal. § 446

Hence the two oblique prisms are equivalent to each other. Q. E. D.

EXERCISES.

408. The four diagonals of a parallelopiped bisect each other.

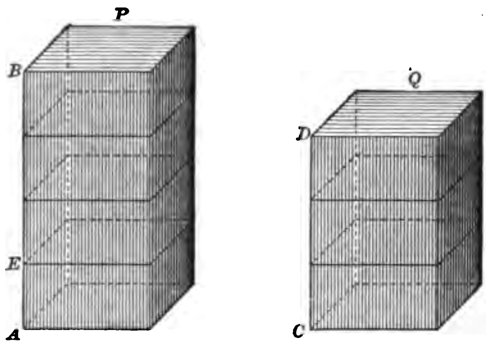
409. The diagonals of a rectangular parallelopiped are equal.

410. The diagonals of an oblique parallelopiped are unequal.

PROPOSITION VIII. THEOREM.

453. ***Two rectangular parallelopipeds having equal bases are to each other as their altitudes.***

CASE I.—*When the altitudes are commensurable.*



Given—P and Q two rectangular parallelopipeds having equal bases, and commensurable altitudes AB and CD.

To Prove— $$\frac{P}{Q} = \frac{AB}{CD}.$$

Dem.—Let AE be a common unit of measure of AB and CD, and suppose it is contained 4 times in AB and 3 times in CD.

Then $$\frac{AB}{CD} = \frac{4}{3}. \qquad (1)$$

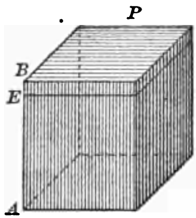
If we pass planes through these points of division parallel to the bases, the parallelopiped P will be divided into four equal parts, and the parallelopiped Q into three equal parts. §§ 442, 446

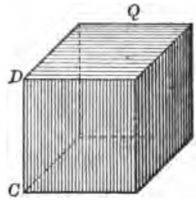
Then $$\frac{P}{Q} = \frac{4}{3}. \qquad (2)$$

And from (1) and (2) we have

$$\frac{P}{Q} = \frac{AB}{CD}. \qquad \text{Ax. 1.}$$

CASE II.—*When the altitudes are incommensurable.*





Given—P and Q two rectangular parallelopipeds having equal bases, and incommensurable altitudes AB and CD.

To Prove— $$\frac{P}{Q} = \frac{AB}{CD}.$$

Dem.—Take any unit of measure that is contained an exact number of times in CD.

Since AB and CD are incommensurable, the unit of measure will be contained in AB a certain number of times, with a remainder EB less than the unit of measure.

Pass a plane through E parallel to the base, and designate the parallelopiped whose altitude is AE by P'.

Then $$\frac{P'}{Q} = \frac{AE}{CD}. \quad \text{Case I.}$$

Now, if we diminish the unit of measure indefinitely, the remainder EB will be diminished indefinitely,

And AE will approach AB as its limit, and P' will approach P as its limit.

Hence we have two variables, $\frac{P'}{Q}$ and $\frac{AE}{CD}$, which are always equal.

And, since the limits of equal variables are equal,

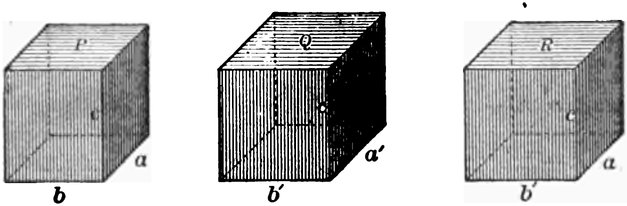
$$\lim. \frac{P'}{Q} = \lim. \frac{AE}{CD}. \quad \S\, 140$$

Therefore $$\frac{P}{Q} = \frac{AB}{CD}. \quad \text{Q. E. D.}$$

454. Cor.—*Two rectangular parallelopipeds having two dimensions the same are to each other as their third dimensions.*

Proposition IX. Theorem.

455. ***Two rectangular parallelopipeds having equal altitudes are to each other as their bases.***



Given—P and Q two rectangular parallelopipeds; the dimensions of P are a, b, c, and those of Q are a', b', c.

To Prove— $$\frac{P}{Q} = \frac{a \times b}{a' \times b'}.$$

Dem.—Construct the rectangular parallelopiped R whose dimensions are a, b', c.

Then the parallelopipeds P and R have the two dimensions a and c the same.

Whence $$\frac{P}{R} = \frac{b}{b'}. \qquad \S\ 454$$

And the parallelopipeds R and Q have the two dimensions b' and c the same.

Whence $$\frac{R}{Q} = \frac{a}{a'}. \qquad \S\ 454$$

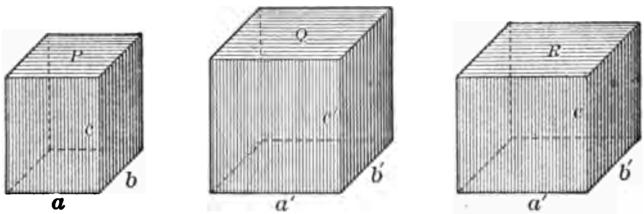
Multiplying these equations, and remembering to cancel R from both terms, we have

$$\frac{P}{Q} = \frac{a \times b}{a' \times b'}. \qquad \text{Q. E. D.}$$

456. COR.—*Two rectangular parallelopipeds having one dimension the same are to each other as the products of their other two dimensions.*

PROPOSITION X. THEOREM.

457. ***Any two rectangular parallelopipeds are to each other as the products of their three dimensions.***



Given—P and Q two rectangular parallelopipeds; the dimensions of P are a, b, c, and those of Q are a', b', c'.

To Prove— $$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Dem.—Construct the rectangular parallelopiped R whose dimensions are a', b', c.

Then the parallelopipeds P and R have equal altitudes.

Whence $$\frac{P}{R} = \frac{a \times b}{a' \times b'}. \qquad \S\,456$$

And the parallelopipeds R and Q have equal bases.

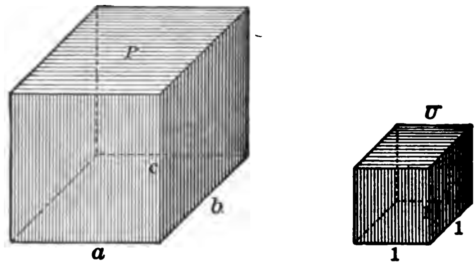
Whence $$\frac{R}{Q} = \frac{c}{c'}. \qquad \S\,454$$

Multiplying these equations, and cancelling R from both terms, we have

$$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$ Q. E. D.

PROPOSITION XI. THEOREM.

458. ***The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being the cube whose edge is the linear unit.***



Given—P a rectangular parallelopiped whose dimensions are a, b, c, and U the unit of volume whose dimensions are 1, 1, 1.

To Prove—Vol. $P = a \times b \times c$.

Dem.—We have $\dfrac{P}{U} = \dfrac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c$. § 457

Since U is the unit of volume,

Then $\dfrac{P}{U} =$ vol. P. § 431

Therefore vol. $P = a \times b \times c$. Q. E. D.

459. COR. 1.—Since $a \times b$ is the area of the base,

Then *the volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

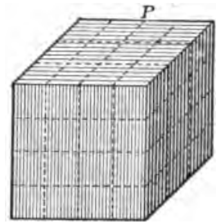
460. COR. 2.—*The volume of a cube is equal to the cube of its edge.*

For, if the three dimensions of a parallelopiped are each equal to a, the solid is a cube whose edge is a.

Hence vol. cube $= a \times a \times a = a^3$.

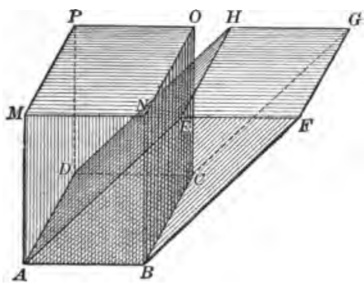
461. Scholium 1.—The volume of a rectangular parallelopiped is, more strictly speaking, equal to $a \times b \times c$ *times* U; that is, *the volume of a parallelopiped is the unit of volume taken as many times as there are units in the product of its three dimensions.*

462. Scholium 2.—When the three dimensions of the parallelopiped are exactly divisible by the linear unit, the proposition is readily proved by dividing the solid into cubes, each of which is equal to the unit of volume.



Proposition XII. Theorem.

463. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Given—$ABCD-E$ any parallelopiped whose base is $ABCD$, and altitude AM.

To Prove—Vol. $ABCD-E = ABCD \times AM$.

Dem.—Construct the rectangular parallelopiped $ABCD-M$ whose base is $ABCD$, and altitude AM.

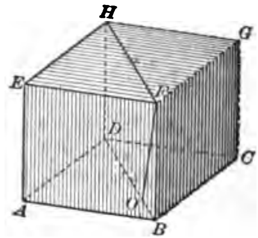
Now, $ABCD-E \Leftrightarrow ABCD-M$. § 451

But vol. $ABCD-M = ABCD \times AM$. § 459

Hence vol. $ABCD-E = ABCD \times AM$. Ax. 1. Q. E. D.

PROPOSITION XIII. THEOREM.

464. ***The volume of a triangular prism is equal to the product of its base and altitude.***



Given—*FO* the altitude of the triangular prism $ABD-E$.

To Prove—Vol. $ABD-E = ABD \times FO$.

Dem.—Complete the parallelopiped $ABCD-E$.

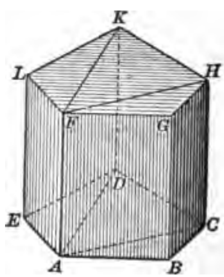
Then vol. $ABD-E = \frac{1}{2}$ vol. $ABCD-E$. § 452

But $\frac{1}{2}$ vol. $ABCD-E = \frac{1}{2} ABCD \times FO$. § 463

Hence vol. $ABD-E = ABD \times FO$. § 80. Q. E. D.

PROPOSITION XIV. THEOREM.

465. ***The volume of any prism is equal to the product of its base and altitude.***



Given—$ABCDE-F$ any prism.

To Prove—Vol. $ABCDE - F = ABCDE \times AF$.

Dem.—Through the lateral edge AF pass the planes AH and AK, dividing the prism into triangular prisms.

Then $\quad$ vol. $ABC - F = ABC \times AF$. $\quad$ § 464

vol. $ACD - F = ACD \times AF$.

vol. $ADE - F = ADE \times AF$.

Adding these equalities, we have

vol. $ABCDE - F = ABCDE \times AF$. $\quad$ Q. E. D.

466. Cor. 1.—*Any two prisms are to each other as the products of their bases and altitudes.*

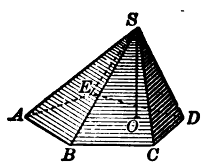
Cor. 2.—*Prisms having equal altitudes are to each other as their bases.*

Cor. 3.—*Prisms having equivalent bases are to each other as their altitudes.*

Cor. 4.—*Prisms having equivalent bases and equal altitudes are equivalent.*

THE PYRAMID.

467. A **Pyramid** is a polyhedron bounded by a polygon and a series of triangular faces having a common vertex; as, $S - ABCDE$.



The polygon $ABCDE$ is called the *base* of the pyramid.

The triangular faces are called the *lateral faces.*

The common vertex of the triangular faces is called the *vertex.*

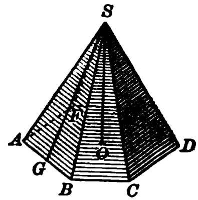
The intersections of the lateral faces are called the *lateral edges.*

The sum of the areas of the lateral faces is called the *lateral area.*

The perpendicular distance SO from the vertex to the plane of the base is called the *altitude.*

468. *Pyramids* are named from their bases. A *Triangular Pyramid* is a pyramid whose base is a triangle; a *Quadrangular Pyramid* is one whose base is a quadrilateral; a *Pentangular Pyramid* is one whose base is a pentagon, etc.

469. A **Regular Pyramid** is a pyramid whose base is a regular polygon (§ 312) and whose vertex lies in the perpendicular erected at the centre of the base.



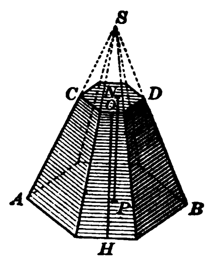
470. The lateral edges of a regular pyramid are equal (§ 377, 2).

Hence the lateral faces are equal isosceles triangles (§ 57).

471. The *Slant Height* of a regular pyramid is the perpendicular distance from the vertex to any side of the base; as, *SG*.

472. A **Truncated Pyramid** is the portion of a pyramid included between its base and a plane cutting all the lateral edges.

473. A **Frustum of a Pyramid** is a truncated pyramid whose bases are parallel.



The *Altitude* of a frustum is the perpendicular distance between its bases; as, *NP*.

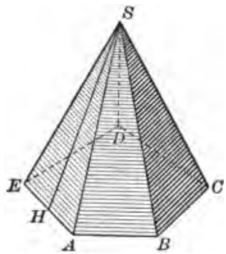
The *Slant Height* of a frustum of a regular pyramid is the altitude of any lateral face; as, *OH*.

The lateral faces of a frustum of a regular pyramid are equal trapezoids.

For the faces of the pyramid are equal (§ 470), and the faces of the pyramid cut off are equal; hence the figures that remain are equal.

PROPOSITION XV. THEOREM.

474. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one half its slant height.*



Given—SH the slant height of the regular pyramid $S-ABCDE$.

To Prove—Lat. area $S-ABCDE = (AB + BC + CD + DE + AE) \times \frac{1}{2}SH$.

Dem.—Since the lateral faces are equal isosceles triangles (§ 470),

Then area $SAE = AE \times \frac{1}{2}SH$, § 231

area $SAB = AB \times \frac{1}{2}SH$, etc.

Adding these equalities, we have

Lat. area $S-ABCDE = (AE + AB + BC + CD + DE) \times \frac{1}{2}SH$. Q. E. D.

475. COR.—*The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of its bases multiplied by the slant height of the frustum.*

EXERCISE.

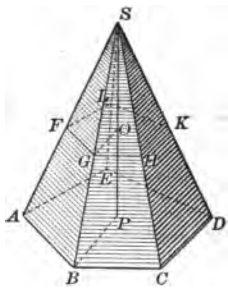
411. Find the lateral area of a regular triangular pyramid whose slant height is 40 ft. and the sides of the base each 30 ft. Of the frustum made by a plane bisecting the slant height.

PROPOSITION XVI. THEOREM.

476. ***If a pyramid is cut by a plane parallel to its base,—***

1st. The edges and altitude are divided proportionally.

2d. The section is a polygon similar to the base.



Given—*FK* a plane parallel to the base of the pyramid *S—ABCDE*, cutting the edges in the points *F*, *G*, *H*, *K*, and *L*, and the altitude in *O*.

To Prove, 1st—$\frac{SA}{SF} = \frac{SB}{SG} = \frac{SC}{SH} = \frac{SD}{SK} = \frac{SE}{SL} = \frac{SP}{SO}$.

Dem.—Since the planes *FK* and *AD* are parallel,

Then *FG* is parallel to *AB*, *GH* to *BC*, etc. § 393

Hence $\frac{SA}{SF} = \frac{SB}{SG} = \frac{SC}{SH} = \frac{SD}{SK} = \frac{SE}{SL} = \frac{SP}{SO}$. § 242

To Prove, 2d—*ABCDE* similar to *FGHKL*.

Dem.—Since *FG* is parallel to *AB*, *GH* to *BC*, etc.,

Then $\angle FGH = \angle ABC$,

$\angle GHK = \angle BCD$, etc. § 398

Hence *ABCDE* and *FGHKL* are mutually equiangular.

Now, $\triangle SAB$ is similar to $\triangle SFG$, $\triangle SBC$ to $\triangle SGH$, etc. § 254

Hence $\frac{FG}{AB} = \left(\frac{SG}{SB}\right) = \frac{GH}{BC} = \left(\frac{SH}{SC}\right) = \frac{HK}{CD}$, etc. § 217

Dropping the ratios in parentheses, the homologous sides of *ABCDE* and *FGHKL* are proportional.

Therefore *ABCDE* is similar to *FGHKL*. § 217. Q. E. D.

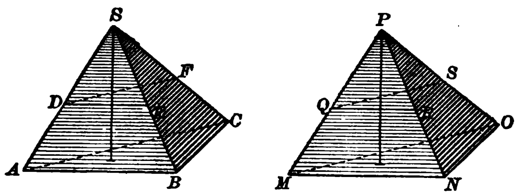
477. Cor. 1.—*The areas of two parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

For $ABCDE : FGHKL = \overline{AB}^2 : \overline{FG}^2$. § 275

But $\overline{AB}^2 : \overline{FG}^2 = \overline{SB}^2 : \overline{SG}^2 = \overline{SP}^2 : \overline{SO}^2$. §§ 254, 128

Hence $ABCDE : FGHKL = \overline{SP}^2 : \overline{SO}^2$. § 131. Q. E. D.

478. Cor. 2.—*If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*



Given—$S-ABC$ and $P-MNO$ two pyramids having equivalent bases, and the altitude of each being H, with the sections *DEF* and *QRS* parallel to the bases and at a distance of h from the vertices.

To Prove—*DEF* equivalent to *QRS*.

Dem.—Since the sections are parallel to the bases,

Then $\frac{DEF}{ABC} = \left(\frac{h^2}{H^2}\right) = \frac{QRS}{MNO}$. § 477

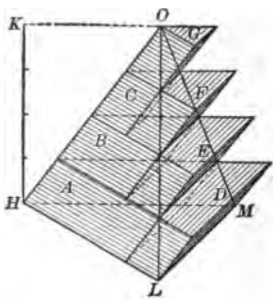
But *ABC* is equivalent to *MNO*.

Therefore *DEF* is equivalent to *QRS*. Q. E. D.

19

PROPOSITION XVII. THEOREM.

479. ***If the altitude of a triangular pyramid be divided into any number of equal parts, and planes be passed through these points of division parallel to the base, and a series of prisms be inscribed in, or circumscribed about, the pyramid, and the number of divisions into which the altitude of the pyramid is divided be indefinitely increased, the limit of the volume of the prisms will be the volume of the pyramid.***



Given—The pyramid $O—HLM$, with the altitude divided into equal parts, and the series of prisms A, B, and C inscribed in, and D, E, F, and G circumscribed about, the pyramid, and the number of equal parts into which the altitude is divided being indefinitely increased.

To Prove—That the limit of the sum of the inscribed or circumscribed prisms will be the volume of the pyramid.

Dem.—Divide the altitude HK into any number of equal parts, as four, and through these points of division pass planes parallel to the base, forming sections in the pyramid.

With these sections as upper bases construct the prisms A, B, C, and denote their sum by s and the lim. s by p.

With these sections as lower bases construct the prisms D, E, F, G, and denote their sum by S and the lim. S by P.

Now, each inscribed prism is equivalent to the circumscribed prism immediately above it. § 466, 4

Hence the difference between the sum of the circumscribed prisms and the sum of the inscribed prisms is the prism D.

Then $S - s = D.$

Now, by increasing the number of parts into which the altitude HK is divided, we can make the volume of D as small as we please, since we diminish the altitude without changing the base.

Then $\lim. (S - s) = \lim. D.$ § 140

But $\lim. (S - s) = P - p.$ § 142

And $\lim. D = 0.$

Hence $P - p = 0.$

And $P = p.$

That is, the limits of the sum of the volumes of the inscribed and circumscribed prisms are equal.

But the volume of the pyramid lies between the inscribed and circumscribed prisms.

Therefore the volume of the pyramid is the limit of the inscribed and circumscribed prisms. Q. E. D.

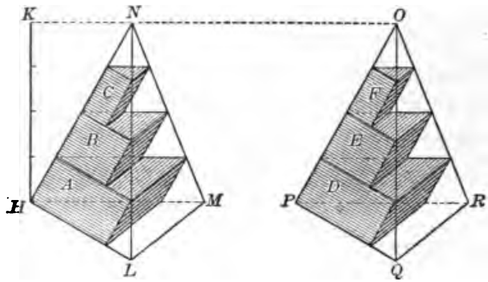
EXERCISES.

412. Each face of a triangular pyramid is an equilateral triangle whose side is 6. Find the entire area.

413. The altitude of a pyramid is 16 ft., and its base is a square 15 ft. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 10 ft.?

PROPOSITION XVIII. THEOREM.

480. ***Two triangular pyramids having equivalent bases and equal altitudes are equivalent.***



Given—$N-HLM$ and $O-PQR$ two pyramids with equal altitudes and equivalent bases.

To Prove— $N-HLM \Leftrightarrow O-PQR$.

Dem.—Place the pyramids so that their bases are in the same plane, and let HK be their common altitude.

Divide HK into any number of equal parts, as four, and through these points of division pass planes parallel to the bases, forming sections in each pyramid.

With these sections as upper bases, construct the prisms A, B, C in $N-HLM$, and D, E, F in $O-PQR$.

The corresponding sections of the two pyramids are equal (§ 478), and, since the altitudes are equal,

$A = D$, $B = E$, and $C = F$. § 466, 4

Let $V = A + B + C$, and $V' = D + E + F$.

Then $V = V'$. Ax. 2.

Now, if the number of parts into which HK is divided is indefinitely increased, V will approach $N-HLM$ as its limit, and V' will approach $O-PQR$ as its limit.

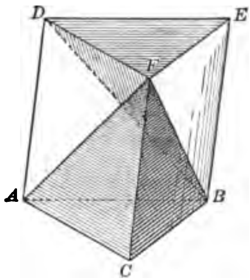
Then lim. $V =$ lim. V'. § 140

But lim. $V = N-HLM$, and lim. $V' = O-PQR$. § 479

Therefore $N-HLM \Leftrightarrow O-PQR$. Q. E. D.

PROPOSITION XIX. THEOREM.

481. ***A triangular pyramid is equivalent to one third of a triangular prism having the same base and altitude.***



Given—$F-ABC$ a triangular pyramid.

On the base ABC construct the prism $ABC-DEF$.

To Prove—Vol. $F-ABC = \frac{1}{3}$ vol. $ABC-DEF$.

Dem.—Pass a plane through the line DF and the point B, forming the triangular pyramids $B-DEF$ and $F-ABD$.

Now, vol. $F-ABC =$ vol. $B-DEF$. § 480

The pyramid $B-DEF$ may be regarded as having the base BED and its vertex at F.

And since the diagonal DB divides the parallelogram $ABED$ into two equal parts, § 80

vol. $F-BED =$ vol. $F-ABD$. § 480

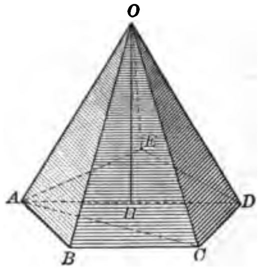
Hence vol. $F-ABC =$ vol. $F-BED =$ vol. $F-ABD$.

Therefore vol. $F-ABC = \frac{1}{3}$ vol. $ABC-DEF$. Q. E. D.

482. COR.—*The volume of a triangular pyramid is equal to one third of the product of its base and altitude.*

PROPOSITION XX. THEOREM.

483. ***The volume of any pyramid is equal to one third of the product of its base and altitude.***



Given—$O-ABCDE$ any pyramid.

To Prove—Vol. $O-ABCDE = \frac{1}{3}$ base $ABCDE \times HO$.

Dem.—Through the lateral edge AO and the diagonals AC and AD (§ 373, 3) pass planes dividing the pyramid into triangular pyramids.

Then vol. $O-ABC = \frac{1}{3}$ base $ABC \times HO$. § 482

vol. $O-ACD = \frac{1}{3}$ base $ACD \times HO$.

vol. $O-ADE = \frac{1}{3}$ base $ADE \times HO$.

Adding these equalities, we have

vol. $O-ABCDE = \frac{1}{3}$ base $ABCDE \times HO$. Q. E. D.

484. COR. 1.—*Any two pyramids are to each other as the products of their bases and altitudes.*

COR. 2.—*Two pyramids having equal altitudes are to each other as their bases.*

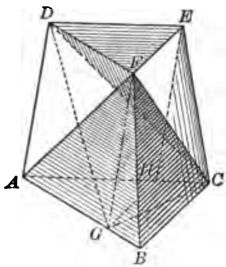
COR. 3.—*Two pyramids having equivalent bases are to each other as their altitudes.*

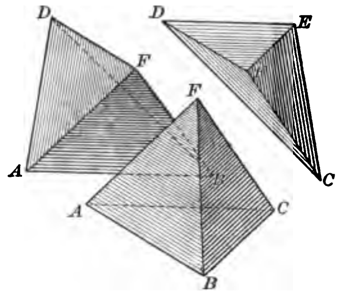
COR. 4.—*Two pyramids having equivalent bases and equal altitudes are equivalent.*

SCHOLIUM.—The volume of any polyhedron may be obtained by dividing it into pyramids.

PROPOSITION XXI. THEOREM.

485. ***The volume of a frustum of a triangular pyramid is equal to the sum of its bases and a mean proportional between them, multiplied by one third of its altitude.***





Given.—$ABC-D$ the frustum of a triangular pyramid.

Denote the area of the lower base by B, the area of the upper base by b, and the altitude by K.

To Prove—Vol. $ABC-D = (B + b + \sqrt{B \times b}) \times \frac{1}{3}K$. § 122

Dem.—Pass a plane through AC and F (§ 373, 1), forming the triangular pyramid $F-ABC$.

Pass a plane through DF and C (§ 373, 1), forming the triangular pyramid $C-DEF$.

There will remain the triangular pyramid $F-ACD$.

Draw FG parallel to AD, then draw GD and GC.

Now, vol. $F-ACD =$ vol. $G-ACD$. § 480

But the pyramid $G-ACD$ may be regarded as having the base AGC and its vertex at D.

Through FE and G pass a plane (§ 373, 1), forming the prism $AGH-DEF$; then $\triangle AGH = \triangle DEF$, § 433

And GH is parallel to FE, and consequently to BC. § 388

Now, $AGH : AGC = AH : AC$, § 232, 3

And $AGC : ABC = AG : AB$. § 232, 3

Since GH is parallel to BC,

$$AH : AC = AG : AB. \qquad \text{§ 242}$$

Hence $AGH : AGC = AGC : ABC$. § 131, Cor.

The vol. $F-ABC = B \times \frac{1}{3}K$. § 482

vol. $C-DEF = b \times \frac{1}{3}K$.

vol. $F-ACD =$ vol. $D-AGC = \sqrt{B \times b} \times \frac{1}{3}K$. § 122

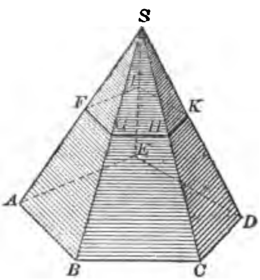
Therefore vol. $ABC-D = (B + b + \sqrt{B \times b}) \times \frac{1}{3}K$.

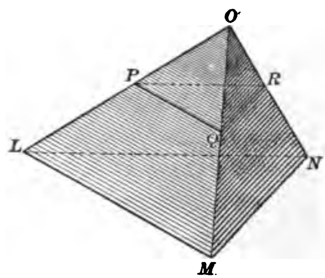
Q. E. D.

Note.—It is seen that $B : AGC = AGC : b$, or $AGC = \sqrt{B \times b}$.

Proposition XXII. Theorem.

486. *The volume of a frustum of any pyramid is equal to the sum of its bases and a mean proportional between them, multiplied by one third of its altitude.*





Given—$ABCDE-F$ the frustum of any pyramid $S-ABCDE$.

Denote its bases by B and b, and its altitude by K.

To Prove—Vol. $ABCDE-F = (B + b + \sqrt{B \times b}) \times \frac{1}{3}K$.

Dem.—Construct the triangular pyramid $O-LMN$, having the same base and altitude as $S-ABCDE$.

Then $S-ABCDE$ is equivalent to $O-LMN$. § 484, 4

Pass a plane through $O-LMN$ the same distance from O as FK is from S, and parallel to the base.

Then PQR is equivalent to $FGHKJ$. § 478

Hence vol. $O-PQR \Leftrightarrow$ vol. $S-FGHKJ$. § 480

Taking away these equivalent pyramids, the remaining frustums are equivalent; they will have equivalent bases and equal altitudes.

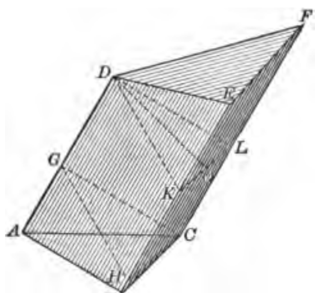
But vol. $LMN-P = (B + b + \sqrt{B \times b}) \times \frac{1}{3}K$. § 485

Hence vol. $ABCDE-F = (B + b + \sqrt{B \times b}) \times \frac{1}{3}K$.

Q. E. D.

Proposition XXIII. Theorem.

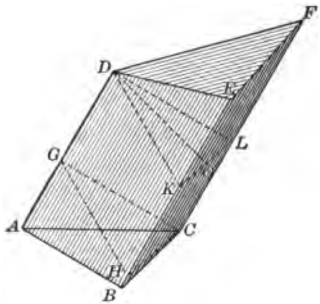
487. *The volume of a truncated triangular prism is equal to the product of a right section by one third of the sum of its lateral edges.*



Given—DKL and GHC right sections of the truncated prism $ABC-DEF$.

To Prove— Vol. $ABC-DEF$

$= DKL \times \frac{1}{3}(AD + BE + CF)$.



Dem.—The prism $ABC-DEF$ is composed of the prism $GHC-DKL$ and the pyramids $D-KLFE$ and $C-ABHG$.

Now, vol. $GHC-DKL = DKL \times GD$. § 464

But $GD = \frac{1}{3}(GD + HK + LC)$.

Hence vol. $GHC-DKL = DKL \times \frac{1}{3}(GD + HK + LC)$. (1)

The vol. $D-KLFE = KLFE \times \frac{1}{3}DO$, § 482

$= \frac{1}{2}(KE + LF) \times KL \times \frac{1}{3}DO$, § 233

$= \frac{1}{2}\, KL \times DO \times \frac{1}{3}(KE + LF)$.

But $\frac{1}{2}KL \times DO =$ area DKL. § 231

Hence vol. $D-KLFE = DKL \times \frac{1}{3}(KE + LF)$. (2)

In like manner,

vol. $C-ABHG = DKL \times \frac{1}{3}(AG + BH)$. (3)

Adding (1), (2), and (3), we have

Vol. $ABC-DEF = DKL \times \frac{1}{3}[(GD + HK + LC) + (KE + LF) + (AG + BH)] = DKL \times \frac{1}{3}(AD + BE + CF)$.

Q. E. D.

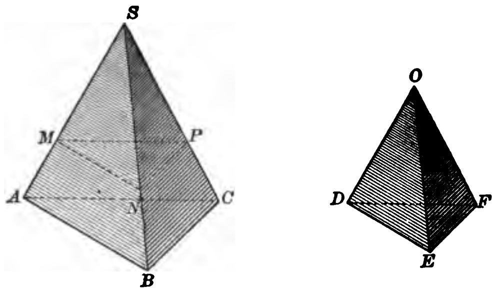
SIMILAR POLYHEDRONS.

488. Similar Polyhedrons are polyhedrons having the same number of faces, similar each to each and similarly placed, and their homologous polyhedral angles equal.

The faces, lines, and angles of similar polyhedrons which are similarly placed are called homologous faces, lines, and angles.

Proposition XXIV. Theorem.

489. *The homologous edges of similar polyhedrons are proportional.*



Given—$S-ABC$ and $O-DEF$ two similar polyhedrons.

To Prove— $\frac{SA}{OD}=\frac{SB}{OE}=\frac{AB}{DE}$, etc.

Dem.—Since the polyhedrons are similar, $O-DEF$ will take the position of $S-MNP$, and their corresponding faces will be similar. § 488

Hence $\frac{SA}{SM}=\frac{SB}{SN}=\frac{AB}{MN}$, etc. § 217

Or $\frac{SA}{OD}=\frac{SB}{OE}=\frac{AB}{DE}$, etc. Q. E. D.

490. Cor. 1.—*Any two homologous faces of two similar polyhedrons are proportional to the squares of any two homologous edges.*

Since the corresponding faces are similar,

Then $\frac{SAB}{ODE}=\frac{\overline{SB}^2}{\overline{OE}^2}=\frac{SBC}{OEF}=\frac{\overline{SC}^2}{\overline{OF}^2}=\frac{SAC}{OFD}$. § 269

491. COR. 2.—*The entire surfaces of two similar polyhedrons are proportional to the squares of any two homologous edges.*

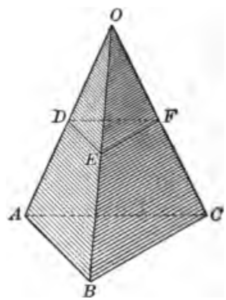
Since $\frac{SAB}{ODE} = \frac{SBC}{OEF} = \frac{SAC}{OFD} = \frac{ABC}{DEF}$. § 490

Then $\frac{SAB + SBC + SAC + ABC}{ODE + OEF + OFD + DEF} = \frac{SAB}{ODE} = \frac{\overline{SB}^2}{\overline{OE}^2}$. § 133

Or surf. $S - ABC$: surf. $O - DEF = \overline{SB}^2 : \overline{OE}^2$.

PROPOSITION XXV. THEOREM.

492. ***If a tetrahedron is cut by a plane parallel to one of its faces, the tetrahedron cut off is similar to the first.***



Given—$DEF - O$ a tetrahedron cut off by the plane DEF parallel to ABC.

To Prove—$DEF - O$ similar to $ABC - O$.

Dem.—The edges AO, BO, and CO are divided proportionally by the plane DEF. § 476

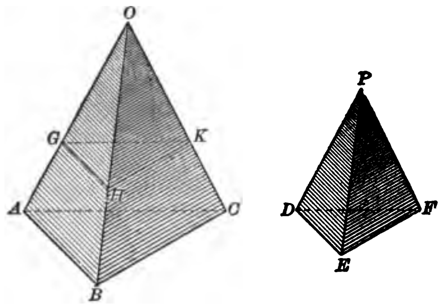
Then DEO is similar to ABO, EFO to BCO, and FDO to ACO. § 254

The homologous trihedral angles are equal, since they are formed by equal face angles. § 425

Therefore $DEF - O$ is similar to $ABC - O$. § 488. Q. E. D.

PROPOSITION XXVI. THEOREM.

493. ***Two tetrahedrons are similar when a dihedral angle of the one is equal to a homologous dihedral angle of the other, and the faces including these angles are similar each to each.***



Given—$ABC-O$ and $DEF-P$ two tetrahedrons, with the dihedral angles OA and PD equal, and the faces ABO and ACO similar to DEP and DFP respectively.

To Prove—$ABC-O$ and $DEF-P$ similar.

Dem.—Since the dihedral angles AO and DP are equal, and the faces including these are similar, the tetrahedron $DEF-P$ will take the position of $GHK-O$.

Since GHO is similar to ABO, and GKO to ACO,

Then $\qquad \frac{GO}{HO} = \frac{AO}{BO}$, and $\frac{GO}{KO} = \frac{AO}{CO}$. § 217

Hence GH is parallel to AB and GK is parallel to AC. § 245

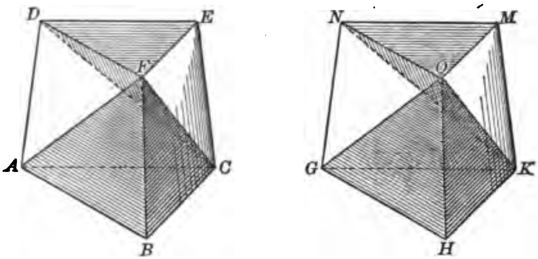
Then the plane GHK is parallel to the plane ABC. § 398

Hence the tetrahedron $GHK-O$ is similar to the tetrahedron $ABC-O$. § 492

Therefore $DEF-P$ is similar to $ABC-O$. Q. E. D.

PROPOSITION XXVII. THEOREM.

494. ***Two similar polyhedrons may be decomposed into the same number of tetrahedrons similar each to each and similarly placed.***



Given—$ABC-F$ and $GHK-O$ similar polyhedrons.

To Prove—That they may be decomposed into the same number of tetrahedrons similar each to each and similarly placed.

Dem.—Draw the diagonals FA, FC, DC, OG, OK, and NK.

Then the polyhedrons will be decomposed into the same number of tetrahedrons, similarly placed.

Let $ABC-F$ and $GHK-O$ be two homologous tetrahedrons.

The triangles ABF and BCF are similar to GHO and HKO respectively. § 272

Since the polyhedrons are similar, the dihedral angle FB is equal to the dihedral angle HO. § 488

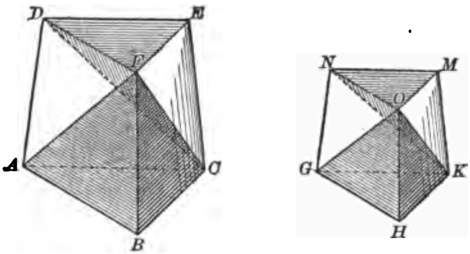
Therefore $ABC-F$ is similar to $GHK-O$. § 493

In like manner, we may prove any two homologous tetrahedrons similar.

Therefore the polyhedrons are decomposed into the same number of tetrahedrons similar each to each and similarly placed. Q. E. D.

Proposition XXVIII. Theorem.

495. ***Two polyhedrons composed of the same number of tetrahedrons similar each to each and similarly placed are similar.***



Given—$ABC-F$ and $GHK-O$ two polyhedrons composed of the same number of tetrahedrons similar each to each and similarly placed.

To Prove—$ABC-F$ similar to $GHK-O$.

Dem.—Since the tetrahedrons are similar, their faces are similar triangles.

Hence $ABFD$ is similar to $GHON$, $ACED$ to $GKMN$, and $BCEF$ to $HKMO$. § 271

Since the tetrahedrons are similar, their trihedral angles are equal.

Hence the polyhedral angle A is equal to the polyhedral angle G, since each angle is composed of two equal trihedral angles.

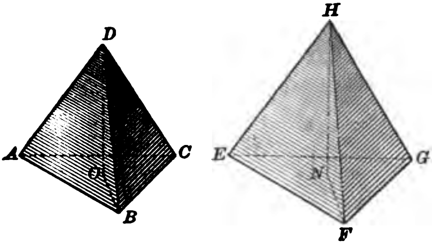
In like manner any other two homologous polyhedral angles may be proved equal.

Therefore $ABC-F$ and $GHK-O$ have equal polyhedral angles similarly placed and the same number of faces similar each to each; hence they are similar.

§ 488. Q. E. D.

PROPOSITION XXIX. THEOREM.

496. ***Two similar tetrahedrons are to each other as the cubes of any two homologous edges.***



Given—$ABC—D$ and $EFG—H$ two similar tetrahedrons, and V and V' their volumes.

To Prove— $\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{EF}^3}$, or $\frac{\overline{DB}^3}{\overline{HF}^3}$.

Dem.—Now, $\frac{V}{V'} = \frac{ABC \times DO}{EFG \times HN}$. § 484

Since the tetrahedrons are similar,

Then $\frac{ABC}{EFG} = \frac{\overline{AB}^2}{\overline{EF}^2}$, § 269

And $\frac{DO}{HN} = \frac{AB}{EF}$. § 217

Multiplying these equalities, $\frac{ABC \times DO}{EFG \times HN} = \frac{\overline{AB}^3}{\overline{EF}^3}$.

Hence $\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{EF}^3}$.

And since any two homologous edges are proportional to any other two, the two similar tetrahedrons are to each other as the cubes of any two homologous edges. Q. E. D.

SCHOLIUM.—Any two similar polyhedrons are to each other as the cubes of their homologous edges.

For any two similar polyhedrons may be decomposed into the same number of tetrahedrons similar each to

each and similarly situated (§ 494). And since the homologous tetrahedrons are to each other as the cubes of their homologous edges (§ 496), then the polyhedrons are to each other as the cubes of their homologous edges.

REGULAR POLYHEDRONS.

497. A **Regular Polyhedron** is a polyhedron whose faces are all equal regular polygons and whose polyhedral angles are all equal.

PROPOSITION XXX. THEOREM.

498. *Only five regular polyhedrons are possible.*

From the definition of a regular polyhedron, the faces must be regular polygons; and at least three faces are necessary to form a polyhedral angle; and from (§ 424), the sum of the face angles of a polyhedral angle must be less than four right angles, or 360°.

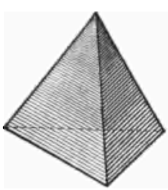
1. *With equilateral triangles only three are possible.*

Since each angle of an equilateral triangle is 60° (§ 101), we may form a polyhedral angle with

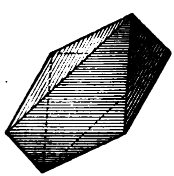
3 triangles, for $3 \times 60° = 180°$, forming a tetrahedron,

4 triangles, for $4 \times 60° = 240°$, forming an octahedron,

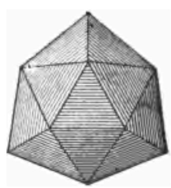
5 triangles, for $5 \times 60° = 300°$, forming an icosahedron.



Tetrahedron



Octahedron

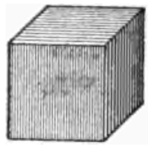


Icosahedron

2. *With squares only one is possible.*

Since each angle of a square is 90°, we may form a polyhedral angle with

3 squares, for $3 \times 90° = 270°$, forming a hexahedron.



Hexahedron or Cube

3. *With regular pentagons only one is possible.*

Since each angle of a regular pentagon is 108° (§ 101), we may form a polyhedral angle with

3 pentagons, for $3 \times 108° = 324°$, forming a dodecahedron.

Dodecahedron

Since each angle of a regular hexagon is 120° (§ 101), no polyhedral angle can be formed with three or more hexagons.

In like manner, no polyhedral angle can be formed with three or more regular polygons of more than six sides; hence these are the only regular polyhedrons.

Therefore only five regular polyhedrons are possible.

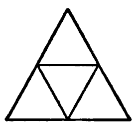
Q. E. D.

499. Scholium.—The five regular polyhedrons may be constructed from cardboard as follows:

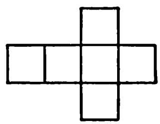
Draw on cardboard the following diagrams.

Cut the figures out entire, and at the lines separating adjacent polygons cut the cardboard half through.

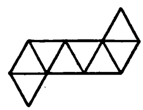
By folding these figures the regular polyhedrons can readily be formed.



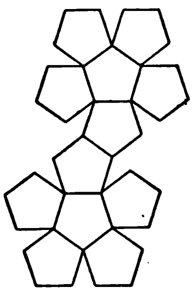
Tetrahedron



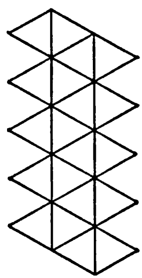
Hexahedron



Octahedron



Dodecahedron



Icosahedron

EXERCISES.

PRACTICAL EXAMPLES.

414. Find the lateral area of a right prism whose base is a regular hexagon 8 inches on a side, and whose altitude is 6 feet.

415. Find the entire area of a right prism whose base is a rectangle 6 inches by 8 inches, and whose altitude is 8 feet.

416. Find the lateral area of a right pyramid whose base is a regular pentagon 10 inches on a side, and whose slant height is 5 yards.

417. Find the entire area of a right pyramid whose base is a regular hexagon 8 inches on a side, and whose slant height is 30 inches.

418. Find the lateral area of the frustum of a right pyramid whose upper base is a regular heptagon 5 inches on a side, the lower base 6 inches, and whose slant height is 18 inches.

419. Find the entire area of the frustum of a right pyramid whose bases are squares, the upper base 6 ft., the lower base 10 ft., and slant height 5 yards.

420. Find the volume of a right hexagonal prism each side of whose base is 8 inches, and whose altitude is 20 inches.

421. Find the volume of the frustum of a square pyramid whose altitude is 24 inches, the sides of the upper base 6 inches, and of the lower base 8 inches.

422. There are two similar triangular pyramids whose homologous sides are 30 and 40 inches respectively; required the relation of their volumes. What is the relation of their areas?

423. Find the edge of a cubical bin that will contain 1200 bushels of grain.

424. The lateral edge of a regular hexagonal pyramid is 10 ft., and the sides of its base 6 ft.; find the lateral area, the entire area, and the volume.

425. Find the lateral area and volume of the frustum of a square pyramid whose altitude is 18 ft., the sides of the lower base 6 ft., and of the upper base 4 ft.

426. Find the volume of a truncated prism whose lateral edges are 12, 16, and 20 ft. respectively, and the sides of a right section are 6, 8, and 10 ft. respectively.

427. How far from the vertex must a square pyramid whose altitude is 12 ft. and base 6 ft. on a side be cut by a plane parallel to the base to divide it into two equivalent parts?

428. Find the edge of a cube whose contents are numerically 10 times its lateral surface.

429. Two bins of similar form contain 360 and 1215 bushels of wheat respectively. If the first bin is 4 ft. long, what is the length of the second?

430. The lateral edge of a frustum of a regular hexagonal pyramid is 12, and the sides of its bases are 12 and 6, respectively; find its lateral area and volume.

431. A square pyramid 40 ft. high, whose base is 12 ft. on a side, is cut by a plane parallel to the base 20 ft. from the vertex; required the area of the upper section.

ORIGINAL THEOREMS.

432. What kind of polygon is formed by projecting the edges of a cube upon a plane perpendicular to a diagonal?

433. The square of any diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three edges meeting at any vertex.

434. The sum of the squares of the four diagonals of a parallelopiped equals the sum of the squares of all the edges.

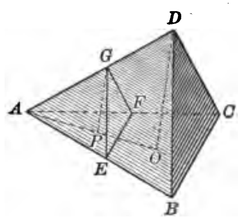
435. Any straight line drawn through the centre and terminating in opposite faces of a parallelopiped is bisected at the centre.

436. The volume of a triangular prism is equal to the product of one of its lateral faces by half the distance of this face from the opposite edge.

437. The lateral area of any pyramid is greater than the area of its base.

438. Two similar right pyramids are to each other as the cubes of their slant heights.

439. Two tetrahedrons having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the edges including the equal trihedral angles.



440. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments which are proportional to the areas of the adjacent faces.

441. The altitude of a pyramid is divided into three equal parts by planes parallel to the base. Find the ratios of the various frustums to one another and to the whole pyramid.

442. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within it upon the four faces.

443. The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.

444. The volume of a regular tetrahedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

445. The volume of a regular octahedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

446. Any two similar pyramids are to each other as the cubes of their homologous edges.

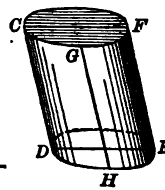
447. Any two similar prisms are to each other as the cubes of their homologous edges.

BOOK VIII.

THE CYLINDER, CONE, AND SPHERE.

THE CYLINDER.

500. A Cylindrical Surface is a curved surface generated by a moving straight line which constantly touches a given curve and in all positions is parallel to a given fixed straight line not in the plane of the curve.



Thus, if the line GH moves so that it constantly touches the curve DHE and in every position is parallel to its original position, the surface generated is a cylindrical surface.

The moving line GH is the *generatrix.*

The curve DHE which it touches is the *directrix.*

Any straight line, as CD or GH, is an *element* of the surface.

In this general definition of a cylindrical surface the directrix may be any curve. The number of kinds of cylindrical surfaces is therefore unlimited. Hereafter we shall assume it a circle, as this is the only curve whose properties are discussed in elementary geometry.

501. A Cylinder is a solid bounded by a cylindrical surface and two parallel planes.

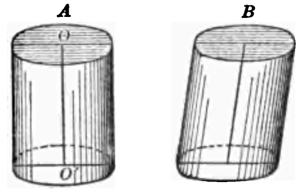
The parallel planes are the *bases.*

The cylindrical surface is the *lateral area.*

The perpendicular distance between the bases is the *altitude.*

COR. It follows from this definition that—*The elements of a cylinder are equal and parallel.* § 394

502. A *Right Cylinder* is a cylinder whose elements are perpendicular to its bases; as *A*.



An *Oblique Cylinder* is one whose elements are oblique to its bases; as *B*.

A *Circular Cylinder* is one whose bases are equal parallel circles.

The *Axis* of a circular cylinder is the line joining the centres of its bases; as OO'.

NOTE.—A right circular cylinder is called a *cylinder of revolution*, since it may be generated by the revolution of a rectangle about one of its sides as an axis.

503. *Similar Cylinders* are cylinders whose axes are proportional to the radii of their bases.

504. A prism is *inscribed in a cylinder* when its bases are inscribed in the bases of the cylinder. The cylinder then *circumscribes* the prism.

A cylinder is *inscribed in a prism* when the bases of the cylinder are inscribed in the bases of the prism. The prism then *circumscribes* the cylinder.

505. A *Section* of a cylinder is a plane figure formed by cutting the cylinder with a plane in any direction.

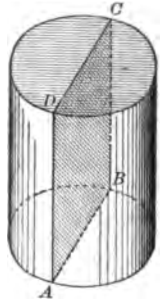
A *Right Section* is a section perpendicular to the axis.

An *Axial Section* is a section formed by a plane passing through the axis.

A plane is *tangent* to a cylinder when it passes through an element of the curved surface without cutting the surface. The element through which it passes is called the *element of contact.*

Proposition I. Theorem.

506. ***A section of a cylinder made by a plane passing through an element is a parallelogram.***



Given—*ABCD* a section of the cylinder *AC* made by a plane passed through the element *AD*.

To Prove—*ABCD* a parallelogram.

Dem.—Pass a plane through *AD*, and it will intersect the circumference in *B*.

Draw *BC* parallel to *AD*. *BC* will then be an element of the surface (§ 500), and will also lie in the plane *AC*. § 31

Now, since *BC* lies in both surfaces, it must be their intersection.

The line *DC* is parallel to *AB*. § 393

Therefore *ABCD* is a parallelogram. § 74. Q. E. D.

507. Cor. 1.—*Every section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

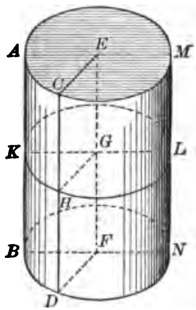
Cor. 2.—*The greatest section of a cylinder made by a plane passing through an element is the axial section.*

Cor. 3.—*The figure formed by joining the corresponding extremities of two elements with straight lines is a parallelogram.*

Cor. 4.—*The elements of a cylinder are equal.*

PROPOSITION II. THEOREM.

508. ***Every section of a circular cylinder made by a plane parallel to its bases is a circle equal to its bases.***



Given—KL a section of the circular cylinder BM parallel to the base BN.

To Prove—KHL a circle equal to the bases.

Dem.—Pass a plane through any element DC and the axis EF.

Then GH and DF are parallel and equal. §§ 393, 81

In like manner it may be proved that the distance from G to any point of the curved line KHL is equal to the radius of the base.

Therefore the section KHL is a circle equal to the base. Q. E. D.

COR. 1.—*The sections of a cylinder made by two parallel planes cutting all its elements are equal.*

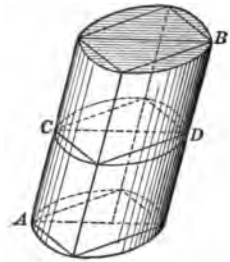
They may be regarded as bases of a cylinder.

COR. 2.—*In a right cylinder the section parallel to the base is perpendicular to the axis.*

SCHOLIUM.—A section formed by a plane perpendicular to the axis of an oblique cylinder is an ellipse.

PROPOSITION III. THEOREM.

509. ***The lateral area of a circular cylinder is equal to the perimeter of a right section, multiplied by an element.***



Given—S the lateral area, C the perimeter of a right section, and E an element, of the circular cylinder AB.

To Prove— $S = C \times E$.

Dem.—Inscribe in the cylinder a prism whose base is a regular polygon.

Denote the lateral area by S', and the perimeter of a right section by P.

Then $S' = P \times E$. § 443

If the number of faces of the prism be indefinitely increased,

Then S' will approach S as its limit, and $P \times E$ will approach $C \times E$ as its limit.

We shall then have two equal variables.

Hence lim. S' = lim. $P \times E$. § 140

Or $S = C \times E$. Q. E. D.

510. COR. 1.—*The lateral area of a right cylinder is equal to the circumference of the base multiplied by the altitude.*

This may be formulated thus, $S = 2\pi R \times H$ (§ 342), in which R is the radius of the base and H the altitude.

511. Cor. 2.—*The total area of a cylinder is*

$$T = 2\pi R(H + R),$$

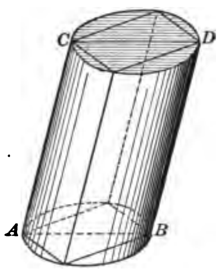
in which T denotes the total area, R the radius of the base, and H the altitude.

For $S = 2\pi R \times H$ (§ 510), and the bases $= 2\pi R^2$. § 344

Adding, $T = 2\pi R(H + R)$.

Proposition IV. Theorem.

512. ***The volume of a circular cylinder is equal to the product of its base and altitude.***



Given—V the volume, B the area of the base, and H the altitude of the circular cylinder AD.

To Prove— $V = B \times H$.

Dem.—Inscribe in the cylinder a prism whose base is a regular polygon.

Denote its volume by V', and the area of the base by B'.

Then $V' = B' \times H$. § 465

If the number of faces of the prism be indefinitely increased,

Then V' will approach V as its limit, and $B' \times H$ will approach $B \times H$ as its limit.

We shall then have two equal variables.

Hence $\text{lim. } V' = \text{lim. } B' \times H$. § 140

Therefore $V = B \times H$. Q. E. D.

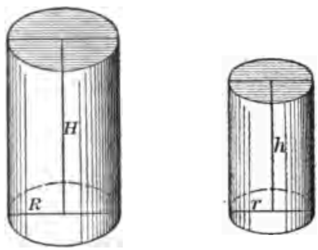
513. Cor.—*The volume of a circular cylinder is*

$$V = \pi R^2 \times H,$$

in which V denotes the volume, R the radius of the base, and H the altitude.

Proposition V. Theorem.

514. *The lateral areas, and also the entire areas, of two similar right circular cylinders are to each other as the squares of their altitudes, or as the squares of the radii of their bases, and their volumes are to each other as the cubes of their altitudes or of their radii.*



Given—S and s the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of any two similar right circular cylinders.

To Prove— $$\frac{S}{s} = \frac{T}{t} = \frac{R^2}{r^2} = \frac{H^2}{h^2},$$

and $$\frac{V}{v} = \frac{R^3}{r^3} = \frac{H^3}{h^3}.$$

Dem.—Since the cylinders are similar, we have the following proportions:

$$\frac{H}{h} = \frac{R}{r} = \frac{H+R}{h+r}. \qquad \text{§§ 503, 125}$$

Then

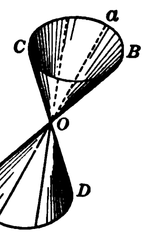
$$\frac{S}{s} = \frac{2\pi R \times H}{2\pi r \times h} = \frac{R}{r} \times \frac{H}{h} = \frac{R^2}{r^2} = \frac{H^2}{h^2}. \qquad \S\ 510$$

$$\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \times \frac{H+R}{h+r} = \frac{R^2}{r^2} = \frac{H^2}{h^2}. \qquad \S\ 511$$

$$\frac{V}{v} = \frac{\pi R^2 \times H}{\pi r^2 \times h} = \frac{R^2}{r^2} \times \frac{H}{h} = \frac{R^3}{r^3} = \frac{H^3}{h^3}. \qquad \text{Q. E. D.}$$

THE CONE.

515. A Conical Surface is a curved surface generated by a moving straight line which constantly touches a given curve and passes through a fixed point not in the plane of the curve.



Thus, if the straight line OA moves so that it constantly touches the curve AFD and in every position passes through the fixed point O, the surface $O—AFD$ is a conical surface.

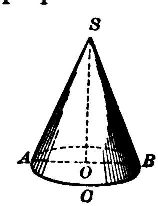
The moving line OA is the *generatrix*.

The curve AFD which it touches is the *directrix*.

Any straight line, as OA or OF, is an *element*.

If the generatrix is of indefinite length, as AOB, the surface generated will consist of two symmetrical parts which are called *nappes*.

In this general definition of a conical surface the directrix may be any curve. Hereafter we shall assume it a circle, as this is the only curve whose properties are discussed in elementary geometry.



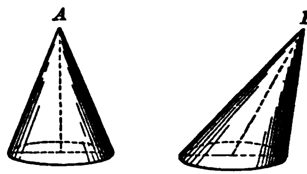
516. A Cone is a solid bounded by a conical surface and a plane cutting the surface; as $S—ACB$. The plane surface ACB is the *base*. The point S is the *vertex*.

The conical surface of a cone is the *lateral area.*

The perpendicular distance SO from the vertex to the base of a cone is the *altitude.*

A *Circular Cone* is a cone whose base is a circle. The straight line drawn from the vertex of a circular cone to the centre of the base is the *axis* of the cone.

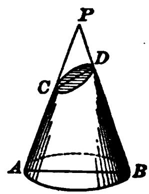
517. A *Right Circular Cone* is a circular cone whose axis is perpendicular to its base; as A.



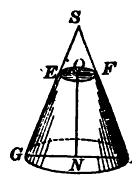
An Oblique Circular Cone is a circular cone whose axis is oblique to its base; as B.

NOTE—A right circular cone is called a *cone of revolution*, since it may be generated by the revolution of a right triangle about one of its legs as its axis.

518. A **Truncated Cone** is the portion of a cone included between its base and a plane cutting its convex surface; as $ABDC$.



519. A **Frustum of a Cone** is a truncated cone whose bases are parallel; as $GHFE$.



The *Altitude* of a frustum of a cone is the perpendicular distance between its bases; as ON.

The *Slant Height* of a frustum of a right circular cone is the portion of an element included between its bases; as EG.

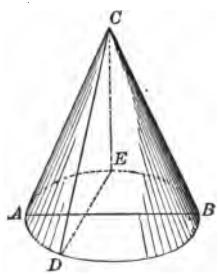
520. *Similar Cones of Revolution* are circular cones whose axes are proportional to the radii of their bases. They may be generated by similar right triangles.

521. A pyramid is *inscribed in a cone* when its base is inscribed in the base of the cone and its vertex coincides with the vertex of the cone. The cone then *circumscribes* the pyramid.

522. A pyramid is *circumscribed about a cone* when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone. The cone is then *inscribed* in the pyramid.

PROPOSITION VI. THEOREM.

523. ***Every section of a cone made by a plane passing through its vertex is a triangle.***



Given—*CDE* a section of a cone made by a plane passing through the vertex *C*.

To Prove—*CDE* a triangle.

Dem.—Pass a plane through *C*, *D*, and *E*.

Draw the straight lines *CD* and *CE*; they are elements of the convex surface, § 515

And *CD* and *CE* lie in the plane *CDE*. § 372

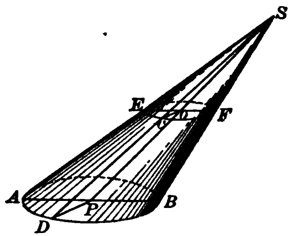
Hence *CD* and *CE* lie in the convex surface and also in the plane, and are therefore their intersections.

Also *DE* is a straight line. § 374

Therefore *CDE* is a triangle. § 41. Q. E. D.

PROPOSITION VII. THEOREM.

524. ***Every section of a circular cone made by a plane parallel to the base is a circle.***



Given—*EF* a section of the circular cone *S—ADB*, parallel to the base *ADB*.

To Prove—*ECF* a circle.

Dem.—Pass planes through *SP* and any elements *SA*, *SD*, etc., cutting the base in *PA*, *PD*, etc., and the parallel section in *OE*, *OC*, etc.

Then *OE* is parallel to *PA*, and *OC* to *PD*. § 393

Hence $\triangle SPA$ is similar to $\triangle SOE$, and $\triangle SPD$ to $\triangle SOC$. § 257

Whence $$\frac{EO}{AP} = \left(\frac{SO}{SP}\right) = \frac{CO}{DP}.$$ § 217

But $AP = DP$; hence $EO = CO$. § 149, 8

Hence all points in the curve *ECF* are equally distant from *O*.

Therefore *ECF* is a circle. Q. E. D.

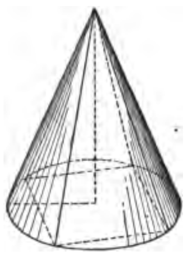
EXERCISES.

448. Sections of a cone parallel to the base are proportional to the square of the distances from the vertex.

449. In cones with equal bases and equal altitudes, sections parallel to the bases and at equal distances from them are equal.

PROPOSITION VIII. THEOREM.

525. *The lateral area of a cone of revolution is equal to the product of the circumference of its base by half its slant height.*



Given—S the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To Prove— $S = C \times \frac{1}{2}L.$

Dem.—Inscribe in the cone a pyramid whose base is a regular polygon.

Denote the lateral area by S', the perimeter of the base by P, and the slant height by L'.

Then $S' = P \times \frac{1}{2}L'.$ § 474

If the number of faces of the pyramid be indefinitely increased,

Then S' will approach S as its limit, P will approach C as its limit, and L' will approach L as its limit.

We shall then have two variable quantities which are always equal.

Hence $\lim. S' = \lim. P \times \frac{1}{2}L'$, § 140

Or $S = C \times \frac{1}{2}L$, Q. E. D.

526. COR. 1.—*The lateral area of a cone of revolution is*

$$S = \pi R \times L.$$

527. COR. 2.—*The total area of a cone of revolution is*

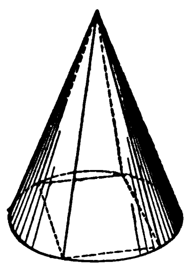
$$T = \pi R(L + R),$$

in which T denotes the total area, R the radius of the base, and L the slant height.

For	$S = \pi R \times L.$	§ 526
The base	$= \pi R^2.$	§ 344
Adding,	$T = \pi R(L + R).$	Q. E. D.

PROPOSITION IX. THEOREM.

528. *The volume of a circular cone is equal to one third of the product of its base and its altitude.*



Given—V the volume, B the area of the base, and H the altitude of the circular cone.

To Prove— $V = \frac{1}{3}B \times H.$

Dem.—Inscribe in the cone a pyramid whose base is a regular polygon.

Denote its volume by V', and the area of its base by B'.

Then $V' = \frac{1}{3}B' \times H.$ § 483

If the number of the faces of the pyramid be indefinitely increased,

Then V' will approach V as its limit, and B' will approach B as its limit.

We shall then have two variables which are always equal.

Hence $\quad\lim. V' = \lim. \frac{1}{3}B' \times H,$ § 140

Or $\quad V = \frac{1}{3}B \times H.$ Q. E. D.

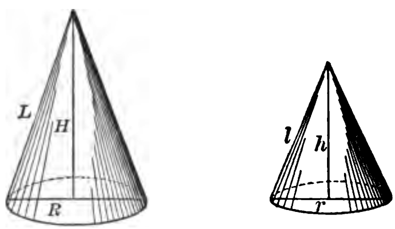
529. COR.—*The volume of a circular cone is*

$$V = \tfrac{1}{3}\pi R^2 \times H,$$

in which R denotes the radius of the base.

PROPOSITION X. THEOREM.

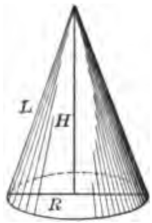
530. ***The lateral areas, and also the entire areas, of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.***

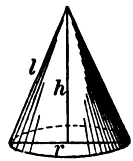


Given—S and s the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases of two similar cones of revolution.

To Prove— $\quad \dfrac{S}{s} = \dfrac{T}{t} = \dfrac{L^2}{l^2} = \dfrac{H^2}{h^2} = \dfrac{R^2}{r^2},$

and $\quad \dfrac{V}{v} = \dfrac{L^3}{l^3} = \dfrac{H^3}{h^3} = \dfrac{R^3}{r^3}.$





Dem.—Since the cones are similar,

$$\frac{L}{l}=\frac{R}{r}=\frac{H}{h}=\frac{L+R}{l+r}. \qquad \S\S\ 520,\ 125$$

Then

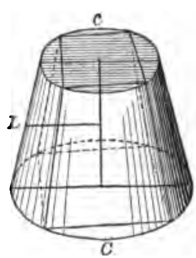
$$\frac{S}{s}=\frac{\pi RL}{\pi rl}=\frac{R}{r}\times\frac{L}{l}=\frac{L^2}{l^2}=\frac{H^2}{h^2}=\frac{R^2}{r^2}. \qquad \S\ 526$$

$$\frac{T}{t}=\frac{\pi R(R+L)}{\pi r(r+l)}=\frac{R}{r}\times\frac{R+L}{r+l}=\frac{L^2}{l^2}=\frac{H^2}{h^2}=\frac{R^2}{r^2}. \qquad \S\ 527$$

$$\frac{V}{v}=\frac{\frac{1}{3}\pi R^2H}{\frac{1}{3}\pi r^2h}=\frac{R^2}{r^2}\times\frac{H}{h}=\frac{L^3}{l^3}=\frac{H^3}{h^3}=\frac{R^3}{r^3}. \qquad \S\ 529. \quad \text{Q. E. D.}$$

PROPOSITION XI. THEOREM.

531. *The lateral area of a frustum of a cone of revolution is equal to one half of the sum of the circumferences of its bases multiplied by its slant height.*



Given—S the lateral area, C and c the circumferences of

the bases, and L the slant height of a frustum of a cone of revolution.

To Prove— $S = \frac{1}{2}(C + c)L.$

Dem.—Inscribe in the frustum the frustum of a pyramid whose base is a regular polygon.

Denote its lateral area by S', the perimeters of its bases by P and p, and the slant height by L'.

Then $S' = \frac{1}{2}(P + p)L'.$ § 475

If the number of the faces of the frustum of the pyramid be indefinitely increased,

Then S' will approach S as its limit, P and p will approach C and c, respectively, as their limits, and L' will approach L as its limit.

We shall then have two variables which are always equal.

Hence $\lim. S' = \lim. \frac{1}{2}(P + p)L',$ § 140

Or $S = \frac{1}{2}(C + c)L.$ Q. E. D.

532. Cor. 1.—*The lateral area of a frustum of a cone of revolution is*

$$S = \pi(R + r)L,$$

in which R and r are the radii of the bases.

For $S = \frac{1}{2}(2\pi R + 2\pi r)L = \pi(R + r)L.$ § 342

533. Cor. 2.—*The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases multiplied by its slant height.*

Let R' be the radius of a section of the frustum equally distant from the bases.

Then $2R' = (R + r).$ § 91

Substituting in the formula of § 532,

$$S = 2\pi R' \times L.$$

Proposition XII. Theorem.

534. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one third its altitude.*

Given—V the volume, B and b the areas of the bases, and H the altitude of a frustum of a right circular cone.

To Prove—$V = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H.$

Dem.—Inscribe in the frustum the frustum of a pyramid whose base is a regular polygon.

Denote its volume by V', and the areas of its bases by B' and b'.

Then $\qquad V' = (B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3}H. \qquad$ § 486

If the number of the faces of the frustum of the pyramid be indefinitely increased,

Then V' will approach V as its limit, and B' and b' will approach B and b, respectively, as their limits.

We shall then have two variables which are always equal.

Hence lim. $V' =$ lim. $(B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3}H, \qquad$ § 140

Or $\qquad V = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H. \qquad$ Q. E. D.

535. *The volume of a frustum of a circular cone is*

$$V = \tfrac{1}{3}\pi(R^2 + r^2 + Rr)H,$$

in which R and r are the radii of the bases.

For $B = \pi R^2$, and $b = \pi r^2$. § 344

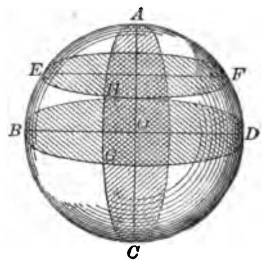
Then $\sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi Rr.$

Substituting in the formula of § 534,

$$V = \tfrac{1}{3}\pi(R^2 + r^2 + Rr)H. \qquad \text{Q. E. D.}$$

THE SPHERE.

536. A **Sphere** is a solid bounded by a surface all points of which are equally distant from a point within, called the *centre*.



A sphere may be generated by the revolution of a semicircle ABC about its diameter AC as an axis.

A *radius* of a sphere is a straight line drawn from the centre to the surface; as OB.

A *diameter* is a straight line drawn through the centre, having its extremities in the surface.

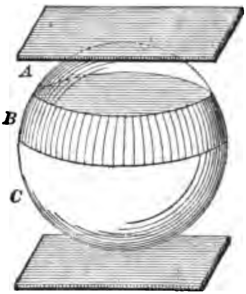
From this definition it is evident that in the same sphere or equal spheres the radii and diameters are equal.

537. A *great circle* is a section made by a plane passing through the centre; as BGD.

A *small circle* is a section made by a plane which does not pass through the centre; as EHF.

The semi-circumference BGD is equal to the semi-circumference ABC, since the radii OB and OA are equal, and is thus one half of the circumference of a great circle.

538. A *Spherical Segment* is a portion of the sphere included between two parallel planes; as *A*, *B*, and *C*.



The sections of the sphere made by the two parallel planes are the *bases* of the segment.

The perpendicular distance between the planes is the *altitude* of the segment.

A *spherical segment of one base* is a spherical segment formed when one of the parallel planes is tangent to the sphere.

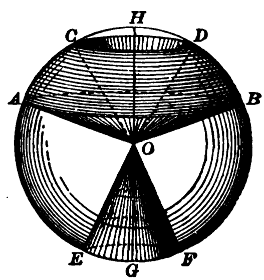
539. The surface of a spherical segment is a *zone*.

The *bases* of the zone are the circumferences of the circles which bound it.

The *altitude* of the zone is the altitude of the corresponding segment.

A *zone of one base* is a zone formed when one of the parallel planes is tangent to the sphere.

540. A *Spherical Sector* is a volume generated by the revolution of a circular sector about the diameter; as *O—ABCD*.

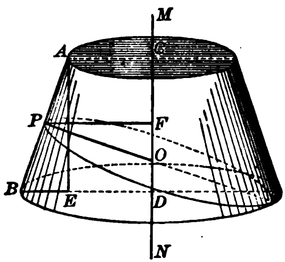


A spherical sector, in general, is bounded by three curved surfaces; namely, the two conical surfaces generated by the radii *OA* and *OC*, and the zone generated by the arc *AC*. The zone is called the base of the spherical sector.

OC may coincide with *OH*, in which case the spherical sector is bounded by one conical surface and a zone of one base; as *O—EGF*.

PROPOSITION XIII. THEOREM.

541. *The area generated by the revolution of a straight line about an axis in the same plane is equal to the projection of the line on the axis multiplied by the circumference of a circle whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.*



Given—The straight line AB revolved about the axis MN in its own plane, CD the projection of AB on MN, and PO the perpendicular erected at the middle point of AB, terminating in the axis.

To Prove—Area $AB = CD \times 2\pi PO$.

Dem.—Draw AE perpendicular to BD, and PF perpendicular to MN.

The area generated by AB is the lateral area of a frustum of a cone of revolution.

Hence	area $AB = AB \times 2\pi PF$.	(1) § 533
But	$\triangle ABE$ is similar to $\triangle POF$.	§ 257
Then	$AB : AE = PO : PF$,	§ 217
And	$AB \times PF = AE \times PO$,	§ 119
	$= CD \times PO$.	

Substituting in (1),

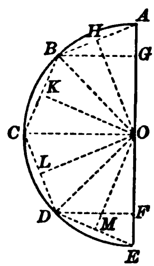
area $AB = CD \times 2\pi PO$. Q. E. D.

542. Cor. 1.—*If AB meets MN, the surface generated is a conical surface whose altitude is CD and the radius of whose base is BD.*

543. Cor. 2.—*If AB is parallel to MN, the surface generated is a cylindrical surface whose radius is BD and altitude CD.*

Proposition XIV. Theorem.

544. ***The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.***



Given—O the centre of the semi-circumference $ABCDE$, with the radius OA equal to R.

To Prove— Sur. sphere $= AE \times 2\pi R$.

Dem.—Inscribe in the semicircle a regular semi-polygon, and revolve both of them about the diameter AE.

Draw OH, OK, OL, and OM perpendicular to the chords AB, BC, CD, and DE; these perpendiculars are equal (§ 163), and bisect the chords (§ 160).

Then area $AB = AG \times 2\pi OH$. § 541

area $BC = GO \times 2\pi OH$.

area $CD = OF \times 2\pi OH$.

area $DE = FE \times 2\pi OH$.

Adding these equations,

area $ABCDE = AE \times 2\pi OH$.

If the number of sides of the semi-polygon be indefinitely increased,

Then the area $ABCDE$ will approach the surface of the sphere, and OH will approach R as its limit.

We shall then have two variables which are always equal.

Hence lim. area $ABCDE =$ lim. $AE \times 2\pi OH$, § 140

Or sur. sphere $= AE \times 2\pi R$. Q. E. D.

545. COR. 1.—*The area of the surface of a sphere is*

$$S = 4\pi R^2 \text{ or } \pi D^2,$$

in which S denotes the surface and R the radius.

546. COR. 2.—*The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

For let S and S' represent the areas of two spheres whose radii are R and R'.

Then $S : S' = 4\pi R^2 : 4\pi R'^2 = R^2 : R'^2$, § 545

And $S : S' = \pi D^2 : \pi D'^2 = D^2 : D'^2$.

547. COR. 3.—*The area of a zone is equal to the circumference of a great circle multiplied by its altitude.*

For the area of the zone generated by BC equals

$$GO \times 2\pi R.$$

548. COR. 4.—*Zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

For let Z and Z' represent the areas of two zones whose altitudes are A and A', on a sphere whose radius is R.

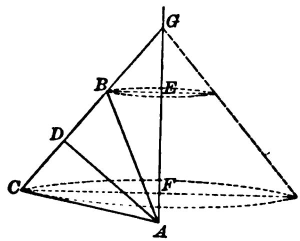
Then $Z : Z' = A \times 2\pi R : A' \times 2\pi R = A : A'$. § 547

549. COR. 5.—*The area of a zone is to the area of the surface of a sphere as the altitude of a zone is to the diameter of the sphere.*

For $Z : S = A \times 2\pi R : D \times 2\pi R = A : D$. §§ 547, 548.

PROPOSITION XV. THEOREM.

550. *If an isosceles triangle be revolved about an axis in the same plane, which passes through its vertex without intersecting its surface, the volume generated is equal to the area generated by the base, multiplied by one third of the altitude.*



Given—The isosceles triangle ABC revolved about the axis FG, and AD its altitude.

To Prove—Vol. $ABC =$ area $CB \times \frac{1}{3}AD$.

Dem.—Produce CB to G, and draw BE and CF perpendicular to AG.

Now, vol. $ACG =$ vol. $ACF +$ vol. GCF,

$$= \tfrac{1}{3}\pi\overline{CF}^2 \times AF + \tfrac{1}{3}\pi\overline{CF}^2 \times GF, \qquad \S\ 529$$

$$= \tfrac{1}{3}\pi\overline{CF}^2 \times (AF + GF) = \tfrac{1}{3}\pi\overline{CF}^2 \times AG,$$

$$= \tfrac{1}{3}\pi CF(CF \times AG).$$

But $CF \times AG = AD \times CG$, each being equal to twice the area of the triangle ACG. § 231

Hence vol. $ACG = \frac{1}{3}\pi CF \times AD \times CG$.

But $\pi CF \times CG$ is the area generated by CG. § 541

Hence vol. $ACG =$ area $CG \times \frac{1}{3}AD$,

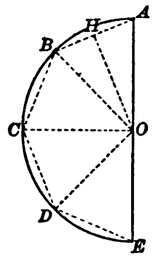
Also vol. $ABG =$ area $BG \times \frac{1}{3}AD$.

Subtracting these two equations,

vol. $ABC =$ area $CB \times \frac{1}{3}AD$. Q. E. D.

PROPOSITION XVI. THEOREM.

551. *The volume of a sphere is equal to the area of its surface multiplied by one third of its radius.*



Given—O the centre of the semi-circumference $ABCDE$, with the radius OA equal to R.

To Prove—Vol. sphere = sur. sphere × $\frac{1}{3}R$.

Dem.—Inscribe in the semicircle a regular semi-polygon, and revolve both of them about the diameter AE.

Draw OB, OC, and OD; also OH perpendicular to AB.

Then OH will bisect the chord AB. § 160

Now, vol. AOB = area $AB \times \frac{1}{3}OH$. § 550

vol. BOC = area $BC \times \frac{1}{3}OH$.

vol. COD = area $CD \times \frac{1}{3}OH$.

vol. DOE = area $DE \times \frac{1}{3}OH$.

Adding, vol. $ABCDE$ = area $ABCDE \times \frac{1}{3}OH$.

If the number of sides of the semi-polygon be indefinitely increased,

Then vol. $ABCDE$ will approach the vol. of the sphere as its limit, the area $ABCDE$ will approach the sur. of a sphere as its limit, and OH will approach R as its limit.

We shall then have two equal variables.

Hence lim. vol. $ABCDE$ = lim. area $ABCDE \times \frac{1}{3}OH$, § 140

Or vol. sphere = sur. sphere × $\frac{1}{3}R$. § 143 Q. E. D.

552. COR. 1.—*The volume of a sphere is*

$$V = \tfrac{4}{3}\pi R^3, \text{ and } V = \tfrac{1}{6}\pi D^3,$$

in which V denotes the volume of the sphere, R the radius, and D the diameter.

Since $V = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3$, § 545

$V = \pi D^2 \times \tfrac{1}{6}D = \tfrac{1}{6}\pi D^3$. § 545

553. COR. 2.—*The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.*

For $V : V' = \tfrac{4}{3}\pi R^3 : \tfrac{4}{3}\pi R'^3 = R^3 : R'^3$. § 552

$V : V' = \tfrac{1}{6}\pi D^3 : \tfrac{1}{6}\pi D'^3 = D^3 : D'^3$. § 552

554. COR. 3.—*The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one third of the radius of the sphere.*

By a process of reasoning entirely analogous to that employed in the theorem, it may be proved that

vol. sph. sector OBD = zone $BD \times \tfrac{1}{3}R$.

555. COR. 4.—*The volume of a spherical sector is equal to two thirds of the area of a great circle, multiplied by the altitude of the zone.*

For area zone $= 2\pi R \times H$. § 547

Then vol. $= 2\pi R \times H \times \tfrac{1}{3}R = \tfrac{2}{3}\pi R^2 \times H$. § 554

556. COR. 5.—*The volumes of spherical sectors on the same or equal spheres are to each other as the zones which form their bases, or as the altitudes of these zones.*

For $V : V' = Z \times \tfrac{1}{3}R : Z' \times \tfrac{1}{3}R = Z : Z'$. § 554

$V : V' = \tfrac{2}{3}\pi R^2 \times H : \tfrac{2}{3}\pi R^2 \times H' = H : H'$. § 555

EXERCISES.

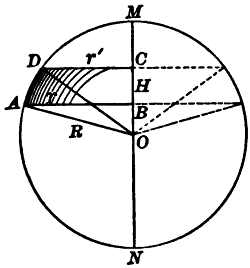
450. Find the surface of a sphere whose radius is 23 in.

451. Find the volume of a sphere whose diameter is 18 in.

452. Find the volume of a spherical sector the altitude of whose base is 8 in., the diameter of the sphere being 18 in.

PROPOSITION XVII. THEOREM.

557. ***To find the volume of a spherical segment.***



Given—O the centre of a sphere whose radius is R,

And $AB = r$, $OB = h'$,

$DC = r'$, $BC = H$,

$OC = h$, vol. $ABCD = V$.

To Prove—The volume of the spherical segment generated by the revolution of $ABCD$ about MN as an axis equal to $\pi(h - h')R^2 - \frac{1}{3}\pi(h^3 - h'^3)$.

Dem.—Draw OA and OD.

Now, seg. $ABCD =$ sect. $AOD +$ cone $DOC -$ cone AOB.

Or $V = \frac{2}{3}\pi R^2 H + \frac{1}{3}\pi r'^2 h - \frac{1}{3}\pi r^2 h'$. §§ 555, 528

But $H = h - h'$, (1)

$$R^2 = r^2 + h'^2 = r'^2 + h^2. \qquad (2)$$

Then $V = \frac{2}{3}\pi(h - h')R^2 + \frac{1}{3}\pi h(R^2 - h^2) - \frac{1}{3}\pi h'(R^2 - h'^2)$.

Or $V = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi R^2 h' + \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi h^3 - \frac{1}{3}\pi R^2 h' + \frac{1}{3}\pi h'^3$.

Collecting, $V = \pi(h - h')R^2 - \frac{1}{3}\pi(h^3 - h'^3)$. (3). Q. E. D.

This formula is convenient when the distances of the bases of the segment from the centre of the sphere are given.

558. COR. 1. *The volume of a spherical segment is equal to the half sum of its bases multiplied by its altitude, plus the volume of a sphere of which that altitude is the diameter.*

Factoring (3), § 557, we have,

$$V = (h - h')[3R^2 - (h^2 + hh' + h'^2)]\frac{\pi}{3}. \quad (4)$$

Squaring (1), $H^2 = h^2 - 2hh' + h'^2.$

Then $hh' = \frac{1}{2}(h^2 + h'^2 - H^2),$

And $h^2 + hh' + h'^2 = h^2 + \frac{1}{2}(h^2 + h'^2 - H^2) + h'^2,$

$$= \tfrac{3}{2}(h^2 + h'^2) - \tfrac{1}{2}H^2,$$

$$= \tfrac{3}{2}(R^2 - r'^2 + R^2 - r^2) - \tfrac{1}{2}H^2,$$

$$= 3R^2 - \tfrac{3}{2}(r^2 + r'^2) - \tfrac{1}{2}H^2.$$

Substituting in (4),

$$V = H[3R^2 - 3R^2 + \tfrac{3}{2}(r^2 + r'^2) + \tfrac{1}{2}H^2]\frac{\pi}{3},$$

$$V = \tfrac{1}{2}H(\pi r^2 + \pi r'^2) + \tfrac{1}{6}\pi H^3.$$

559. COR. 2.—*The volume of a spherical segment of one base is*

$$\tfrac{1}{2}\pi r^2 H + \tfrac{1}{6}\pi H^3.$$

EXERCISES.

ORIGINAL EXAMPLES.

453. Find the lateral area and volume of a cylinder whose altitude is 40 in. and radius of the base 10 in.

454. Find the lateral area and volume of a cone whose altitude is 8 yards and radius of the base 4 ft.

455. Find the entire area of the frustum of a cone whose slant height is 12 ft., the radius of the upper base 3 ft., and of the lower base 6 ft.

456. Find the volume of the frustum of a cone whose altitude is 12 in., the radius of the upper base 8 in., and of the lower base 12 in.

457. Find the surface and volume of a sphere whose radius is 16 in.

458. Find the area of a zone whose altitude is 4 in., on a sphere whose diameter is 20 in.

459. Find the volume of a spherical sector, the altitude of the zone which forms its base being 6 in., on a sphere whose radius is 15 in.

460. Find the volume of a spherical segment of one base; the altitude of the zone which forms its base is 8 in., and the diameter of the sphere 24 in.

461. Find the volume of a spherical segment, the radii of whose bases are 12 in. and $2\sqrt{22}$ in. and whose altitude is 4 in.

462. Find the volume of the largest cube that can be cut from a sphere 12 in. in diameter.

463. Find the volume of the sphere circumscribing a cube whose volume is 512 cu. ft.

464. If a sphere 8 in. in diameter weighs 729 pounds, what is the diameter of a sphere of the same material whose weight is 1331 pounds?

465. Find the diameter of a sphere whose volume and surface are numerically equal.

466. If the diameter of a sphere is 30 inches, find the entire surface of a circumscribing cylinder.

467. What is the ratio numerically of the surface to the volume of a sphere whose diameter is 12 in.?

468. If a sphere has a diameter of 60 ft., what is the surface of a zone of one base whose altitude is 15 ft.?

469. A small circle whose radius is 6 ft. is 8 ft. from the centre of a sphere; find the volume of the sphere.

470. The radius of a sphere is 12 in.; find the area of a small circle 4 in. from the centre.

471. The altitude of a zone is 4 ft. and the radius of the sphere is 10 ft.; find the area of the zone and the volume of the corresponding spherical sector.

472. A cone whose slant height is equal to the diameter of its base is circumscribed about a sphere whose radius is 6 ft.; find the surface and volume of the cone.

473. The edge of a cube is 16 in.; what is the volume of the inscribed sphere?

474. The diameter of a sphere is 12 in.; what is the volume of the circumscribing cylinder?

475. The area of the entire surface of a frustum of a cone of revolution is 268π sq. in., and the radii of its bases are 6 in. and 8 in.; find its lateral area and volume.

476. The volume of a frustum of a cone of revolution is 151π cu. in., its altitude is 3 in., and the radius of its lower base is 9 in.; find the radius of its upper base, its slant height, and its lateral area.

ORIGINAL THEOREMS.

477. What is the relation of the surfaces of a cube and an inscribed sphere?

478. What is the relation of the volumes of a cube and an inscribed sphere?

479. What is the relation of the surfaces of a cube and a circumscribed sphere?

480. What is the relation of the volumes of a cube and a circumscribed sphere?

481. What is the relation of the volumes of two cylinders generated by successively revolving a rectangle about its two adjacent sides?

482. Prove that the surface of a sphere is equal to the lateral area of a circumscribed cylinder.

483. Prove that the surface of a sphere is to the entire area of a circumscribed cylinder, including its bases, as 2 to 3.

484. Prove that the volumes of a cone of revolution, a sphere, and a cylinder of revolution are in the proportion of 1, 2, 3 if the bases of the cone and cylinder are each equal to a great circle of the sphere and their altitudes are each equal to the diameter of the sphere.

485. Prove that the sum of the squares of three chords at right angles to one another from any point in the surface of a sphere is equal to the square of the diameter.

486. The entire area of a cylinder of revolution whose altitude and diameter are equal is a mean proportional between the surface of the circumscribed sphere, and the entire area of a cone of revolution whose slant height and diameter are equal, and which is inscribed in the same sphere. The same is true of the volumes of these three bodies.

487. The entire area of a cylinder of revolution whose altitude and diameter are equal is a mean proportional between the surface of the inscribed sphere, and the entire area of a cone of revolution whose slant height and diameter are equal, and which is circumscribed about the same sphere. The same is true of the volumes of these three bodies.

488. A sphere $2R$ inches in diameter is bored through the centre with a $2r$-inch auger; derive a formula for finding the volume of the part remaining.

489. If an equilateral triangle is revolved about its altitude, find the relation of the solids generated by the triangle, the circumscribed circle, and the inscribed circle.

490. How far must a person ascend above the earth to see one fourth of its surface?

BOOK IX.

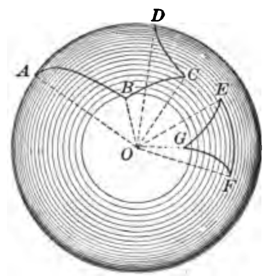
SPHERICAL GEOMETRY.

SPHERICAL SURFACES.

560. A **Spherical Angle** is the angle between two intersecting arcs of great circles.

The arcs are the *sides* of the angle. The point at which the two arcs meet is called the *vertex* of the angle. The measure of a spherical angle is the same as the measure of the dihedral angle formed by the planes of its sides.

561. A **Spherical Polygon** is a portion of the surface of a sphere bounded by three or more arcs of *great circles*; as *ABCD*, or *EFG*.



The diagonal of a spherical polygon is the arc of a *great circle* connecting any two vertices which are not consecutive.

562. A **Spherical Triangle** is a spherical polygon of three sides.

Spherical triangles are *equilateral*, *isosceles*, etc., in the same manner as plane triangles.

563. A **Lune** is the portion of the surface of a sphere bounded by the semi-circumferences of two *great circles*.

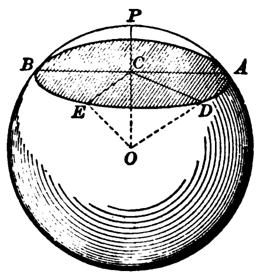
A **Spherical Wedge** is the solid bounded by a lune and the planes of its bounding arcs.

564. A **Spherical Pyramid** is the solid bounded by a spherical polygon and the planes of its sides; as *O—ABCD*, or *O—EFG.*

565. A **Pole of a Circle** is a point on the surface of a sphere equally distant from every point in the circumference of the circle.

PROPOSITION I. THEOREM.

566. *Every section of a sphere made by a plane is a circle.*



Given—*ADB* a section of a sphere made by a plane.

To Prove—*ABED* a circle.

Dem.—Draw *OP* perpendicular to the plane of *ADB*.

Take *E* and *D* any two points in the perimeter of *ADB*, and draw *OE* and *OD*.

Then $OE = OD$. § 536

Hence $CE = CD$. § 378

But *E* and *D* are any two points in the perimeter *ADB*.

Hence *ADB* is a circle whose centre is *C*. Q. E. D.

567. COR. 1.—*A diameter drawn perpendicular to the plane of a small circle passes through its centre.*

568. COR. 2.—*All great circles on the same sphere are equal.*

569. COR. 3.—*Every great circle divides a sphere into two equal parts.*

570. Cor. 4.—*Any two great circles bisect each other.*

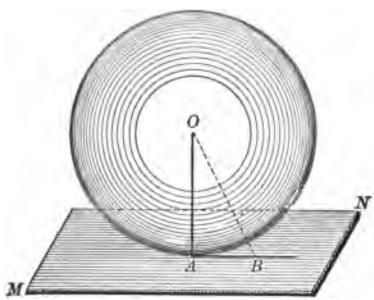
571. Cor. 5.—*An arc of a great circle may be drawn through any two given points of the surface of a sphere, and, if the points are not opposite extremities of a diameter, only one arc can be drawn.*

For the two given points and the centre determine the plane of a great circle. § 373

572. Cor. 6.—*A circumference of a circle may be passed through any three given points on the surface of a sphere.*

Proposition II. Theorem.

573. ***A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.***



Given—MN a plane perpendicular to the radius OA at its extremity A.

To Prove—MN tangent to the sphere.

Dem.—From the centre O draw OB to any point of the plane MN, except A.

Then $OB > OA$. § 377, 1

Hence every point of the plane, except A, lies without the sphere.

Therefore MN is tangent to the sphere. Q. E. D.

574. Cor. 1.—*A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Given—MN tangent to the sphere at A.

To Prove—MN perpendicular to OA.

Draw OB to any point of the plane MN, except A.

Since the plane MN is tangent to the sphere, every point of AB, except A, lies without the sphere.

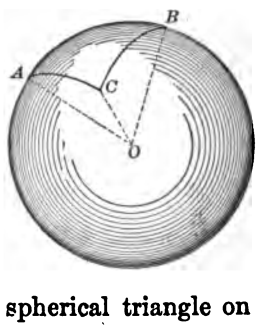
Hence OA is the shortest line that can be drawn from the point O to the plane MN.

Therefore OA is perpendicular to the plane MN.

575. Cor. 2.—*A line perpendicular to a tangent plane at its point of contact passes through the centre of the sphere.*

Proposition III. Theorem.

576. ***Any side of a spherical triangle is less than the sum of the other two sides.***



Given—ABC a spherical triangle on the sphere whose centre is O.

To Prove— $AC + BC > AB$.

Dem.—Draw OA, OB, and OC.

Then $\angle AOC + \angle COB > \angle AOB$. § 423

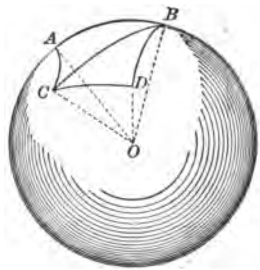
But the arcs AC, CB, and AB are the measures of the angles AOC, COB, and AOB, respectively. § 177

Therefore $AC + BC > AB$.

In like manner

$AC + AB > BC$, and $AB + BC > AC$. Q. E. D.

577. COR. 1.—*Any side of a spherical polygon is less than the sum of the other sides.*



Draw the diagonal CB.

Then $AB < AC + CB$. § 576

But $CB < CD + DB$.

Hence $AB < AC + CD + DB$.

578. COR. 2.—*The shortest distance from one point to another on the surface of a sphere is measured on the arc of a great circle.*

For, suppose the two points to be connected by the arc of a small circle. Divide this arc into any number of equal parts, and through these points of division draw arcs of great circles, thus forming a spherical polygon. The sum of these arcs will be greater than the arc of a great circle joining the two given points (§ 577). But if the number of divisions be indefinitely increased, the limit of the sum of these arcs will be the arc of the small circle joining the two given points, and will be greater than the arc of a great circle joining the same points.

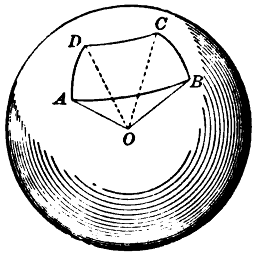
EXERCISES.

491. One and only one plane can be passed through a given point on a sphere tangent to that sphere.

492. Sections of a sphere at equal distances from the centre of a sphere are equal.

PROPOSITION IV. THEOREM.

579. ***The sum of the sides of a spherical polygon is less than the circumference of a great circle.***



Given—$ABCD$ a spherical polygon on the sphere whose centre is O.

To Prove—$AB + BC + CD + DA <$ the circumference of a great circle.

Dem.—Draw the radii AO, BO, CO, and DO.

Then $\angle AOB + \angle BOC + \angle COD + \angle DOA < 4$ rt. angles. § 424

But the arcs AB, BC, CD, and DA are the measures of the angles AOB, BOC, COD, and DOA. § 177

Hence the arcs $AB + BC + CD + DA < 4$ rt. angles.

But four right angles is the measure of a circumference.

Therefore $AB + BC + CD + DA <$ the circumference of a great circle. Q. E. D.

EXERCISES.

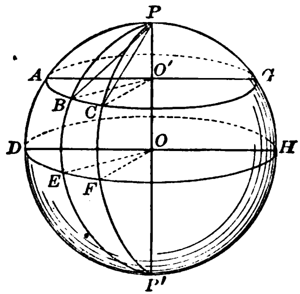
493. The sections of a sphere are proportional to the products of the segments of the diameter drawn perpendicular to them.

494. Find the locus of the centres of spherical sections made by planes that contain a given straight line.

495. Find the locus of the centres of spherical sections which pass through a given point.

PROPOSITION V. THEOREM.

580. ***If a diameter of a sphere is drawn perpendicular to the plane of any circle of a sphere, its extremities are poles of that circle.***



Given—O the centre of a sphere, ABC any circle of a sphere, and PP' a diameter of the sphere perpendicular to the plane of ABC.

To Prove—P and P' poles of the circle ABC.

Dem.—The diameter PP', drawn perpendicular to the plane of ABC, passes through the centre O. § 567

Draw the arcs of great circles PCP' and PBP', and the chords PC and PB.

Then chord $PC =$ chord PB. § 377

Hence arc $PC =$ arc PB. § 155

But B and C are any points of the circumference of the circle ABC.

Therefore P is the pole of the circle ABC, § 565

Since $PCP' = PBP'$, § 570

And arc $PC =$ arc PB.

Subtracting, arc $P'B =$ arc $P'C$.

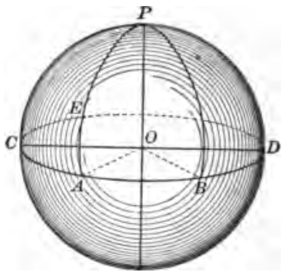
Therefore P' is a pole of the circle ABC. § 565. Q. E. D.

581. Cor. 1.—*The poles of a great circle are at equal distances from the circumference, and those of a small circle are at unequal distances.*

582. Cor. 2.—*The polar distance of a great circle is a quadrant.*

Proposition VI. Theorem.

583. ***If a point on the surface of a sphere is a quadrant's distance from each of two points in the arc of a great circle, it is the pole of that arc.***



Given—P a point on the surface of a sphere CD, at a quadrant's distance from each of the points A and B.

To Prove—P the pole of the arc of a great circle AB.

Dem.—Draw the radii OA, OB, and OP.

Then, since the arcs PA and PB are quadrants, the angles POA and POB are right angles.

Then PO is perpendicular to the plane AOB. § 380

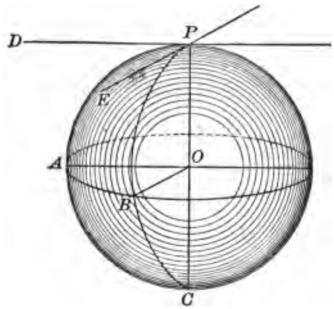
Therefore P is a pole of the arc AB. § 580. Q. E. D.

584. Cor.—*If AE is perpendicular to the arc CAB, it passes through the pole D.*

Note.—This theorem is not always true. If the two points lie at the extremities of a diameter, any number of great circles may be drawn through them, but P is the pole of only one of them.

Proposition VII. Theorem.

585. ***A spherical angle is measured by the arc of a great circle described with its vertex as a pole, included between its sides produced if necessary.***



Given—*APB* a spherical angle on the sphere whose centre is *O*, and *AB* the arc of a great circle described from *P* as a pole.

To Prove—That *AB* is the measure of $\angle APB$.

Dem.—Draw the tangents *PD* and *PE*, and the radii *AO*, *BO*, and *PO*. Then $\angle DPE = \angle AOB$. §§ 169, 403

Since *AB* is the arc of a great circle described from *P* as a pole, *PA* and *PB* are quadrants (§ 582) and *AO* and *BO* are perpendicular to *PO*.

Then $\angle AOB$ is the plane angle of the dihedral angle *PO*.

Hence $\angle AOB = \angle DPE = \angle APB$. § 403, 1

But $\angle AOB$ is measured by the arc *AB*. § 177

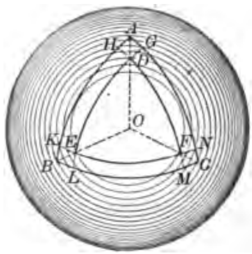
Therefore *AB* is the measure of $\angle APB$. Q. E. D.

586. Cor. 1.—*A spherical angle is equal to the plane angle formed by the tangents to the arcs at their point of intersection.*

587. Cor. 2.—*The opposite or vertical angles formed by two arcs of great circles intersecting each other are equal.*

PROPOSITION VIII. THEOREM.

588. *If from the vertices of a spherical triangle as poles, arcs be described forming a second spherical triangle, the vertices of the second triangle are respectively poles of the sides of the first.*



Given—*ABC* a spherical triangle, and *DEF* a second triangle described from *A*, *B*, and *C* as poles.

To Prove—*D*, *E*, and *F* respectively poles of the arcs *BC*, *AC*, and *AB*.

Dem.—Since *B* is a pole of the arc *DF*, the distance from *D* to *B* is a quadrant. § 582

Since *C* is a pole of the arc *DE*, the distance from *D* to *C* is a quadrant. § 582

Then *D* is a quadrant's distance from the points *B* and *C*.

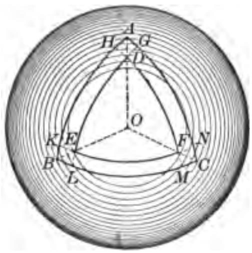
Hence *D* is a pole of the arc *BC*. § 583

In like manner *F* may be shown to be the pole of the arc *AB*, and *E* the pole of the arc *AC*. Q. E. D.

589. SCHOLIUM.—The triangle *ABC* may be described with *D*, *E*, and *F* as poles, as *DEF* is described with *A*, *B*, and *C* as poles. Triangles so related are called **Polar Triangles.**

Proposition IX. Theorem.

590. ***Any angle in one of two polar triangles is measured by a semi-circumference minus the side lying opposite to it in the other triangle.***



Given—ABC and DEF two polar triangles.

To Prove—

$\angle A = 180°-EF$, $\angle B = 180°-DF$, $\angle C = 180°-DE$,

$\angle D = 180°-BC$, $\angle E = 180°-AC$, $\angle F = 180°-AB$.

Dem.—Produce the sides of the triangle DEF to meet the sides of the triangle ABC.

Since E is the pole of the arc AC, and F of the arc AB, then EN and KF are quadrants. § 582

Then $\quad$ arc $EN +$ arc $KF = 180°$.

But $\quad$ arc $EN +$ arc $KF =$ arc $KN +$ arc EF,

Or $\quad$ arc $KN +$ arc $EF = 180°$,

And $\quad$ arc $KN = 180° -$ arc EF.

But, since A is the pole of the arc KN, $\angle A$ is measured by the arc KN. § 585

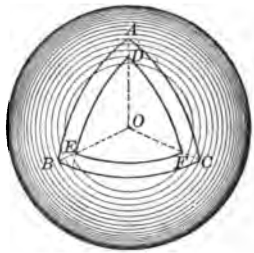
Therefore $\quad \angle A = 180° -$ arc EF.

In like manner the theorem may be proved for any other angle in either triangle. Q. E. D.

591. Cor.—*Each angle of a spherical triangle is less than two right angles.*

PROPOSITION X. THEOREM.

592. ***The sum of the angles of a spherical triangle is less than six, and greater than two, right angles.***



Given—ABC a spherical triangle.

To Prove—$A+B+C<540°$ and $>180°$.

Dem.—1st,

$$\left.\begin{array}{l} A<180°. \\ B<180°. \\ C<180°. \end{array}\right\} \qquad \S\,591$$

Hence $$A+B+C<540°.$$

2d,

$$\left.\begin{array}{l} A=180°-EF. \\ B=180°-DF. \\ C=180°-DE. \end{array}\right\} \qquad \S\,590$$

Hence $$A+B+C=540°-(EF+DF+DE).$$

But $$EF+DF+DE<360°. \qquad \S\,579$$

Therefore $$A+B+C>180°. \qquad \text{Q. E. D.}$$

593. COR.—*A spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.*

DEF.—A spherical triangle having—

One right angle is called a *rectangular* triangle,

Two right angles is called a *bi-rectangular* triangle,

Three right angles is called a *tri-rectangular* triangle.

Each side of a tri-rectangular triangle is a quadrant, and is also called a *tri-quadrantal.*

594. *The surface of a sphere is eight times the surface of a tri-rectangular triangle.*

595. Scholium.—In measuring the angles of a spherical triangle the right angle is taken as the unit, or 1.

The *Spherical Excess* of a spherical triangle is the excess of the sum of the angles above two right angles.

Thus, if we denote the spherical excess by E,

$$E = A + B + C - 180°.$$

The *Spherical Excess* of a spherical polygon is the excess of the sum of the angles above two right angles taken as many times as the polygon has sides, less two.

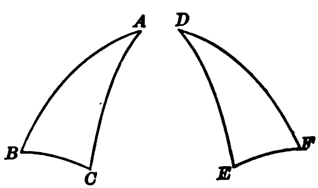
If we denote the number of sides of the polygon by n, and the sum of the angles by S, then

$$E = [S - 2(n-2)] \text{ right angles.}$$

EQUAL AND SYMMETRICAL TRIANGLES.

596. *Equal spherical triangles* are spherical triangles which being applied to each other coincide in all their parts.

597. *Symmetrical spherical triangles* are spherical triangles in which all the parts of one triangle are respectively equal to the parts of the other, but the corresponding parts are arranged in opposite order in the two triangles.



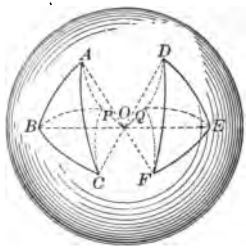
Thus, let ABC and DEF be two spherical triangles in which the sides and angles are equal, but arranged in inverse order; these triangles are symmetrical.

Two symmetrical triangles are not capable of superposition unless they are *isosceles.*

598. *Two isosceles symmetrical triangles are equal.* For they may be made to coincide throughout.

PROPOSITION XI. THEOREM.

599. ***Two symmetrical spherical triangles are equivalent.***



Given—ABC and DEF two symmetrical triangles having $AB = DE$, $AC = DF$, and $BC = FE$.

To Prove— $\triangle ABC \Leftrightarrow \triangle DEF$.

Dem.—Suppose P and Q the poles of small circles which pass through A, B, C and D, E, F.

Then these circles will be equal.

For the chords AC, AB, and BC will equal DF, DE, and FE respectively. § 156

Hence plane $\triangle ABC =$ plane $\triangle DEF$, § 57

And the circumscribed circles are equal.

Then the arcs PA, PB, PC, QD, QE, QF are all equal. § 565

Hence the triangles ACP and DFQ are mutually equilateral, and also isosceles.

Then $\triangle ACP = \triangle DFQ$, § 598

$\triangle ABP = \triangle DEQ$,

And $\triangle BCP = \triangle EFQ$.

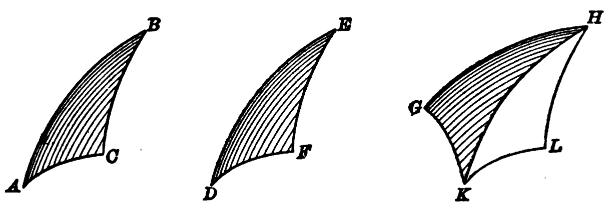
Whence $ABP + BCP - ACP = DEQ + FEQ - DFQ$,

Or $\triangle ABC \Leftrightarrow \triangle DEF$. Q. E. D.

Note.—In this proposition, and in those which follow, the two triangles may be on the same, or equal, spheres.

Proposition XII. Theorem.

600. ***Two spherical triangles on the same sphere are equal or equivalent when two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.***



I. **Given**—ABC and DEF two spherical triangles, on the same sphere, having $AB = DE$, $AC = DF$, $\angle A = \angle D$, and the parts of the two triangles in the same order.

To Prove— $\triangle ABC = \triangle DEF$.

Dem.—Place $\triangle ABC$ upon $\triangle DEF$ so that AB will coincide with DE; and, since $\angle A = \angle D$, AC will take the direction of DF, and C will fall on F; and, since only one arc of a great circle can be drawn connecting any two points (§ 571), CB will coincide with FE.

Hence $\triangle ABC = \triangle DEF$.

II. **Given**—ABC and GHK two spherical triangles, on equal spheres, having $AB = HK$, $AC = GK$, $\angle A = \angle K$, and the parts of the two triangles situated in an inverse order.

To Prove— $\triangle ABC \Leftrightarrow \triangle GKH.$

Dem.—Construct $\triangle HKL$ symmetrical with $\triangle GHK$; then all the parts of $\triangle HKL$ are equal respectively to the parts of $\triangle GHK$. § 597

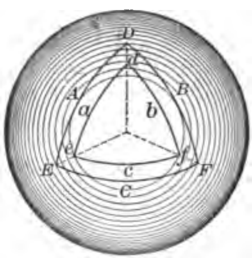
Then $\triangle ABC = \triangle HKL,$ First part.

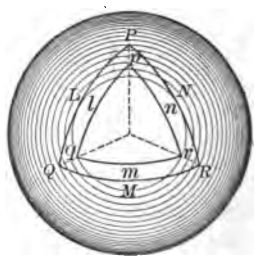
And $\triangle HKL \Leftrightarrow \triangle GHK.$ § 599

Hence $\triangle ABC \Leftrightarrow \triangle GHK.$ Q. E. D.

Proposition XIII. Theorem.

601. ***Two spherical triangles on the same sphere are equal or equivalent when two angles and an included side of the one are respectively equal to two angles and an included side of the other.***





Given—*DEF* and *PQR* two spherical triangles on equal spheres, with $\angle D = \angle P$, $\angle E = \angle Q$, and $A = L$.

To Prove—$\triangle DEF = \triangle PQR$; or $\triangle DEF \Leftrightarrow \triangle PQR$.

Dem.—Construct the polar triangles of *DEF* and *PQR*.

Then $c = m$, $b = n$, and $\angle f = \angle r$. § 590

Hence $\triangle def = \triangle pqr$, or $\triangle def \Leftrightarrow \triangle pqr$. § 600

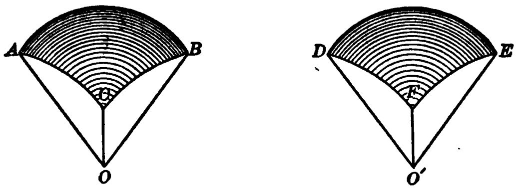
Then their polar triangles will be equal, § 590

And $\triangle DEF = \triangle PQR$, or $\triangle DEF \Leftrightarrow \triangle PQR$. Q. E. D.

602. Scholium.—This proposition can be proved by direct superposition, as in § 56. Compare also § 600.

PROPOSITION XIV. THEOREM.

603. ***If two spherical triangles on the same sphere are mutually equilateral, they are mutually equiangular.***



Given—ABC and DEF two mutually equilateral spherical triangles on equal spheres.

To Prove—The triangles ABC and DEF mutually equiangular.

Dem.—Draw the radii OA, OB, OC, $O'D$, $O'E$, and $O'F$.

Then the homologous face angles of the trihedral angles $O-ABC$ and $O'-DEF$ are equal. § 177

Hence dihedral angle $OA =$ dihedral angle $O'D$. § 425

But the spherical angles BAC and EDF are equal to the dihedral angles OA and $O'D$. § 560

Therefore $\angle BAC = \angle EDF$.

In like manner, $\angle ABC = \angle DEF$,

And $\angle ACB = \angle DFE$.

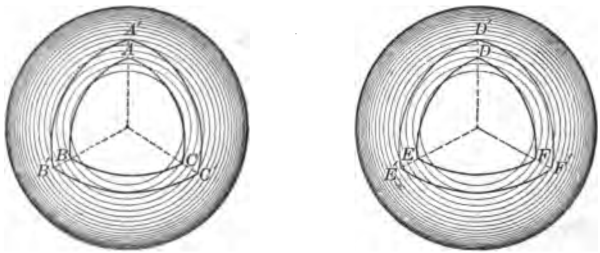
Hence the triangles ABC and DEF are mutually equiangular. Q. E. D.

604. COR.—*Two mutually equilateral spherical triangles on the same sphere are equal or equivalent.*

The triangles are equal if their parts are arranged in the same order, and symmetrical and equivalent if their parts are arranged in an inverse order.

PROPOSITION XV. THEOREM.

605. ***If two spherical triangles on the same sphere are mutually equiangular, they are mutually equilateral.***



Given—ABC and DEF two mutually equiangular spherical triangles on equal spheres.

To Prove—ABC and DEF mutually equilateral.

Dem.—Construct the polar triangles of ABC and DEF.

Then, since ABC and DEF are mutually equiangular, $A'B'C'$ and $D'E'F'$ are mutually equilateral. § 590

Hence $A'B'C'$ and $D'E'F'$ are mutually equiangular. § 604

Then their polar triangles ABC and DEF are mutually equilateral. § 590. Q. E. D.

606. COR.—*Two mutually equiangular spherical triangles on the same sphere are equal or equivalent.*

EXERCISES.

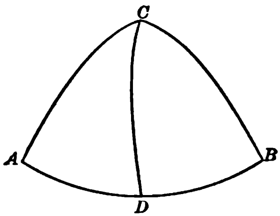
496. If the sides of a polar triangle are 80°, 150°, and 163° respectively, how many degrees are there in each angle of its polar triangle?

497. If the angles of a spherical triangle are 84°, 120°, and 70° respectively, how many degrees are there in each side of its polar triangle?

498. How far above the earth's surface must a person ascend to see ⅓ of its surface?

PROPOSITION XVI. THEOREM.

607. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Given—ABC an isosceles spherical triangle, with $AC = BC$.

To Prove— $\angle A = \angle B$.

Dem.—Draw the arc of a great circle from C to the middle of the base AB.

Then the $\triangle$'s ADC and DBC are mutually equilateral.

Hence they are mutually equiangular. § 603

Therefore $\angle A = \angle B$. Q. E. D.

608. COR.—*The arc of a great circle drawn from the vertex of an isosceles triangle to the middle of the base bisects the vertical angle and is perpendicular to the base.*

EXERCISES.

499. The greater the distance of the plane of a small circle from the centre of the sphere, the less the circle.

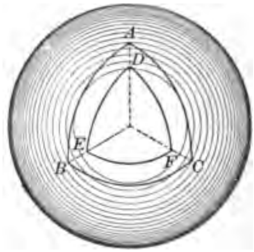
500. Circles whose centres are equidistant from the centre of a sphere are equal.

501. Prove that the radius of a small circle is less than the radius of the sphere.

502. Every point on the arc of a great circle that bisects a spherical angle is equally distant from the sides of the angle; any point not in the bisector is unequally distant from the sides.

PROPOSITION XVII. THEOREM.

609. ***If two angles of a spherical triangle are equal, the sides opposite them are equal.***



Given—ABC a spherical triangle, with $\angle B = \angle C$.

To Prove— $AB = AC$.

Dem.—Construct the polar triangle of ABC.

Then DEF will have $DE = DF$, § 590

And $\angle E = \angle F$. § 607

Then in the polar triangle of DEF, or ABC,

$AB = AC$. § 590. Q. E. D.

EXERCISES.

503. An equiangular spherical triangle is also equilateral.

504. An equilateral spherical triangle is also equiangular.

505. A circle may be inscribed in a spherical triangle.

506. Either angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

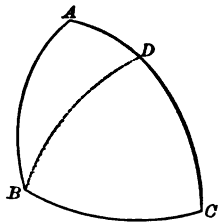
507. The intersection of two spheres is a circle whose plane is perpendicular to the straight line joining the centres of the spheres, and whose centre is in that line.

508. If the arc of a great circle is drawn perpendicular to the arc of another great circle at its middle point, a point in the perpendicular will be equally distant from the extremities of the second arc.

509. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

PROPOSITION XVIII. THEOREM.

610. ***In any spherical triangle the greater side lies opposite the greater angle.***



Given—ABC a spherical triangle, with $\angle B > \angle C$.

To Prove— $AC > AB$.

Dem.—Draw the arc of a great circle BD, making $\angle DBC = \angle C$.

Then	$BD = DC$.	§ 609
But	$AD + BD > AB$.	§ 576
Hence	$AD + DC > AB$,	
Or	$AC > AB$.	Q. E. D.

611. COR.—*In any spherical triangle the greater angle lies opposite the greater side.*

Given—ABC a spherical triangle, with the side $AC > AB$.

To Prove— $\angle B > \angle C$.

Dem.—If $\angle B = \angle C$, AC would equal AB. § 576

If $\angle B < \angle C$, AC would be less than AB. § 610

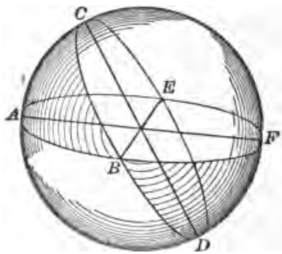
But each of these conclusions is contrary to the hypothesis that $AC > AB$.

Therefore $\angle B > \angle C$.

MEASUREMENT OF SPHERICAL POLYGONS.

PROPOSITION XIX. THEOREM.

612. ***If the arcs of great circles intersect on the surface of a hemisphere, the sum of the opposite spherical triangles which they form is equivalent to a lune whose angle is the angle formed by the circumferences.***



Given—*ABF* and *CBD* two great circles intersecting on the hemisphere *ACFD*.

To Prove—$\triangle ABC + \triangle BDF \Leftrightarrow$ lune *BDEFB*.

Dem.—Produce the arcs *ABF* and *CBD* until they meet in *E*.

The semi-circumference $ABF =$ the semi-circumference *BFE*.

Subtracting the common arc *BF*,

Then arc $AB =$ arc EF. Ax. 2

In like manner it may be proved that arc $CB =$ arc ED, and arc $AC =$ arc FD.

Therefore $\triangle ABC \Leftrightarrow \triangle DEF$. § 603

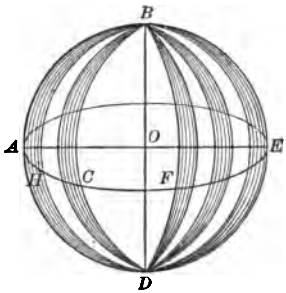
But $\triangle BDF + \triangle DEF =$ lune *BDEFB*.

Hence $\triangle ABC + \triangle BDF \Leftrightarrow$ lune *BDEFB*. Q. E. D.

PROPOSITION XX. THEOREM.

613. ***Two lunes on the same sphere are to each other as their angles.***

CASE I.—*When the angles are commensurable.*



Given—*ABCDA* and *FBEDF* two lunes, whose angles *ABC* and *FBE* are commensurable.

To Prove— $$\frac{ABCDA}{FBEDF} = \frac{\angle ABC}{\angle FBE}.$$

Dem.—Let *ABH* be a common unit of measure of the angles *ABC* and *FBE*, and suppose it is contained 3 times in *ABC* and 5 times in *FBE*,

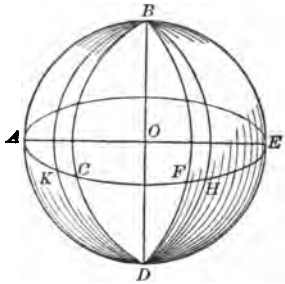
Then $$\frac{\angle ABC}{\angle FBE} = \tfrac{3}{5}.$$

If we draw arcs of great circles through these points of division, the lune *ABCDA* will be divided into 3 parts and *FBEDF* into 5 parts; these parts will be equal, since each part is composed of two equal isosceles triangles.

Then $$\frac{ABCDA}{FBEDF} = \tfrac{3}{5}.$$

Hence $$\frac{ABCDA}{FBEDF} = \frac{\angle ABC}{\angle FBE}. \qquad \text{Ax. 1}$$

CASE II.—*When the angles are incommensurable.*



Given—*ABCDA* and *FBEDF* two lunes, whose angles *ABC* and *FBE* are incommensurable.

To Prove— $$\frac{ABCDA}{FBEDF} = \frac{\angle ABC}{\angle FBE}.$$

Dem.—Take any angle *FBH* as a unit of measure which is contained an exact number of times in $\angle$ *FBE.*

Then, since $\angle$ *ABC* and $\angle$ *FBE* are incommensurable, $\angle$ *FBH* will be contained in $\angle$ *ABC* a certain number of times, with a remainder *KBC* which is less than the unit of measure.

Produce *BK* to *D.*

Since the angles *ABK* and *FBE* are commensurable, we have, by Case I.,

$$\frac{ABKDA}{FBEDF} = \frac{\angle ABK}{\angle FBE}.$$

Now, if we diminish the unit of measure indefinitely, the remainder *KCB* will diminish indefinitely,

And $\angle$ *ABK* will approach $\angle$ *ABC* as its limit,

And *ABKDA* will approach *ABCDA* as its limit.

Hence we have two variables which are always equal.

And, since the limits of equal variables are equal,

Then $\quad \lim. \dfrac{ABKDA}{FBEDF} = \lim. \dfrac{\angle ABK}{\angle FBE},$ § 140

Or $\quad \dfrac{ABCDA}{FBEDF} = \dfrac{\angle ABC}{\angle FBE}.$ Q. E. D.

614. Cor. 1.—*The surface of a lune is to the surface of a sphere as the angle of the lune is to four right angles.*

615. Cor. 2.—*The area of a lune is*

$$L = 2A \times T,$$

in which L denotes the area of the lune, A its angle (expressed in right angles), and T the area of a tri-rectangular triangle.

The surface of a sphere is $8T$. § 594

Hence $\quad \dfrac{L}{8T} = \dfrac{A}{4},$

Or $\quad L = 2A \times T.$

616. Cor. 3.—*The volume of a spherical wedge is to the volume of a sphere as the angle of the wedge is to four right angles.*

This may be demonstrated by a process of reasoning entirely analogous to that employed in the theorem.

617. Cor. 4.—*The volume of a spherical wedge is equal to the volume of a tri-rectangular pyramid multiplied by twice the angle of the wedge.*

EXERCISES.

510. Two lunes on the same sphere or on equal spheres are to each other as their angles.

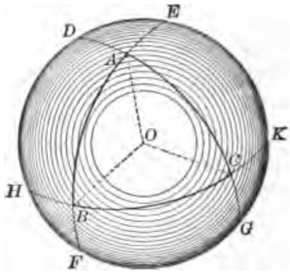
511. Find the area of a tri-rectangular triangle on a sphere whose diameter is 24 inches.

512. The angle of a lune is 30° on a sphere whose radius is 48 ft.; find the surface of the lune.

513. Find the volume of a tri-rectangular spherical pyramid, the diameter of the sphere being 8 in.

PROPOSITION XXI. THEOREM.

618. ***The area of a spherical triangle is equal to its spherical excess multiplied by a tri-rectangular triangle.***



Given—ABC a spherical triangle on a sphere whose centre is O.

To Prove—Area $ABC = (A + B + C - 2) \times T$.

Dem.—Produce the sides till they meet the great circle HGE.

Then $\triangle DAE + \triangle FAG =$ a lune whose angle is A. § 612

But a lune whose angle is A $= 2A \times T$. § 615

In like manner $\triangle HBF + \triangle EBK = 2B \times T$,

And also $\triangle GCK + \triangle HCD = 2C \times T$.

Adding these six triangles, we observe that their sum exceeds the surface of a hemisphere, or $4T$, by twice the area of ABC.

Hence $2 \times \triangle ABC = (2A + 2B + 2C) \times T - 4T$,

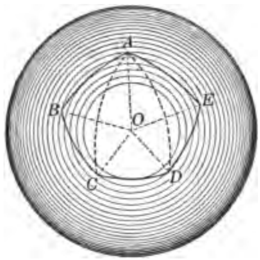
Or $\triangle ABC = (A + B + C - 2) \times T$. Q. E. D.

EXERCISE.

514. Find the area of a spherical triangle whose angles are 90°, 110°, and 120°, on a sphere whose radius is 60 ft. Of one whose angles are each 120°.

PROPOSITION XXII. THEOREM.

619. *The area of a spherical polygon is equal to its spherical excess multiplied by a tri-rectangular triangle.*



Given—$ABCDE$ a spherical polygon of n sides, on a sphere whose centre is O, the sum of whose angles is S, and whose area is H.

To Prove— $H = (S - 2n + 4) \times T$.

Dem.—Divide the polygon into triangles by drawing the diagonals.

Then there will be $n - 2$ triangles.

The area of each triangle is equal to its spherical excess multiplied by a tri-rectangular triangle. § 618

Hence area $ABC =$ (sum of angles $- 2) \times T$.

area $ACD =$ (sum of angles $- 2) \times T$, etc.

Since the sum of the angles of the triangles is equal to the sum of the angles of the polygon, the sum of the areas of the triangles is equal to the area of the polygon, and the spherical excess is equal to

$$S - 2(n - 2). \qquad \S\ 295$$

Then, by adding these equalities,

$$\text{area } ABCDE = [S - 2(n - 2)] \times T,$$

Or $H = (S - 2n + 4) \times T$. Q. E. D.

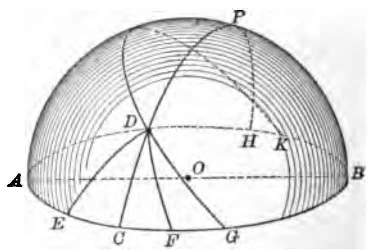
PROPOSITION XXIII. THEOREM.

620. *If from any point on a hemisphere two arcs of a great circle are drawn perpendicular to the circumference of the hemisphere, and oblique arcs are drawn, then,—*

1. The shorter of the two perpendicular arcs is the shortest arc that can be drawn from the given point to the circumference; and the longer of the two perpendicular arcs is the longest arc that can be drawn from the given point to the circumference.

2. Any two oblique arcs which cut off equal distances from the foot of the perpendicular are equal.

3. The oblique arc which cuts off the greater distance from the foot of the perpendicular is the greater arc.



Given—D a point on a hemisphere whose centre is O, DC and DH two arcs perpendicular to the circumference ACB, DE and DF oblique arcs cutting off equal distances from C, and DG cutting off a greater distance from C than DE,

To Prove, 1st—DC the shortest distance from D to the circumference ACB, and DH the longest distance.

Dem.—Let DF be any oblique arc drawn from D to ACB.

Then in the triangle DCF, DC is less than DF. § 610

Hence DC is the shortest distance to ACB.

To prove DH the longest distance from D to the circumference ACB,

Let FDK be any other great circle than CDH, passing through D.

Then arc $FDK =$ arc CDH,

And arc $FD >$ arc CD. 1st part.

Subtracting, arc $DK <$ arc DH.

To Prove, 2d— arc $DE =$ arc DF.

Dem.—The two triangles DCE and DCF have two sides and the included angle of each respectively equal, and are therefore symmetrical. § 600

Hence arc $DE =$ arc DF. § 597

To Prove, 3d— arc $DG >$ arc DE.

Dem.—Take $DF = DE$.

Then in the triangle DFG we have

arc $DG >$ arc DF, or its equal DE. § 610. Q. E. D.

EXERCISES.

515. Find the area of a spherical triangle whose angles are 100°, 120°, and 140°, respectively, on a sphere whose radius is 36 in.

516. Find the volume of a triangular spherical pyramid the angles of whose base are 86°, 100°, and 124°, respectively, and the diameter of the sphere is 24 ft.

517. Find the area of a spherical polygon whose angles are 96°, 110°, 120°, 124°, and 140°, on a sphere whose radius is 12 ft.

518. Find the area of a spherical quadrilateral, the angles being 100°, 110°, 120°, and 130°, the diameter of the sphere being 12 ft.

519. Find the volume of a spherical segment, the radius of the sphere being 20 ft., and the planes of the bases being 8 and 12 ft. from the centre of the sphere.

520. What is the volume of a spherical wedge the angle of whose base is 120°, if the volume of the sphere is 144 cu. ft.?

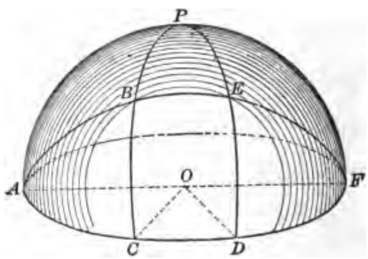
PROPOSITION XXIV. THEOREM.

621. ***If the sides of a spherical angle be produced until they meet, and thus form a lune, then,—***

1st. If the included angle is acute, the longest arc of a great circle that can be drawn between its sides, perpendicular to either side, is the measure of the angle.

2d. If the included angle is obtuse, the shortest arc of a great circle that can be drawn between its sides, perpendicular to either side, is the measure of the angle.

CASE I.—*When the angle is acute.*



Given—*BAC* an acute spherical angle on a sphere whose centre is *O*, with its sides produced forming the lune *ACFBA*, and *DE* its measure.

To Prove—*DE* the longest arc of a great circle that can be drawn between *ADF* and *AEF* perpendicular to *ADF*.

Dem.—Since *DE* is the measure of ∠*A*, *A* is the pole of the arc *DE*, and *DE* is perpendicular to *ADF* and *AEF*. § 585

Produce *DE*, and it will pass through the pole *P* of the arc *ADF* (§ 584), which lies without the sides, since ∠*A* is acute.

Draw any other arc of a great circle, as *BC*, perpendicular to *ADF*, it will also pass through the pole *P* of the arc *ADF*. § 584

Since *AED* is a right angle, *AEP* is also a right angle.

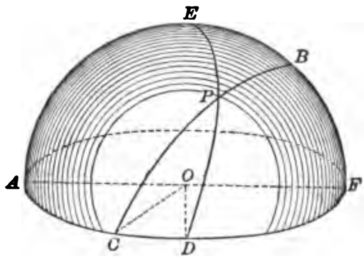
Then arc $PE <$ arc PB. § 610

Now, subtracting *PE* and *PB* from *PD* and *PC*, respectively,

$$\text{arc } DE > \text{arc } BC.$$

Therefore *DE* is the longest arc that can be drawn between *ADF* and *AEF*, perpendicular to *ADF*.

CASE II.—*When the angle is obtuse.*



Given—*BAC* an obtuse spherical angle on a sphere whose centre is *O*, with its sides produced forming the lune *ACFBA*, and *DE* its measure.

To Prove—*DE* the shortest arc of a great circle that can be drawn between *ADF* and *AEF* perpendicular to *ADF*.

Dem.—Since *DE* is the measure of $\angle A$, *A* is a pole of the arc *DE*, and *DE* is perpendicular to *ADF* and *AEF*. § 585

Now, *DE* passes through the pole *P* of the arc *ADF* (§ 584), which lies within the sides, since $\angle A$ is obtuse.

Draw any other arc of a great circle, as *BC*, perpendicular to *ADF*, it will also pass through the pole *P* of the arc *ADF*. § 584

Since AEP is a right angle, PEB is also a right angle.

Then arc $PE <$ arc PB. § 610

Now adding PE and PB to PD and PC, respectively,

$$\text{arc } DE < \text{arc } BC.$$

Therefore DE is the shortest arc that can be drawn between ADF and AEF perpendicular to ADF. Q. E. D.

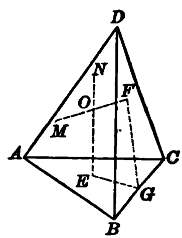
EXERCISES.

ORIGINAL THEOREMS.

521. The area of a spherical triangle, each of whose angles is 120°, is equal to the area of a great circle.

522. A sphere can be circumscribed about any tetrahedron.

Suggestion.—All points equally distant from A, B, and C lie in the perpendicular EN erected at the centre of the circle passing through A, B, and C. In like manner all points equally distant from B, C, and D lie in the perpendicular MF erected at the centre of the circle passing through B, C, and D. Also EN and FM lie in the plane perpendicular to BC at its middle point, etc.



523. A sphere can be inscribed in any tetrahedron.

524. The four perpendiculars erected at the centres of the faces of a tetrahedron meet in the same point.

525. The six planes perpendicular to the six edges of a tetrahedron at their middle point will intersect in a common point.

526. The six planes bisecting the six dihedral angles of a tetrahedron intersect in a common point.

527. The volume of a spherical wedge is to the volume of the sphere as the angle of the lune which forms its base is to four right angles.

528. The volume of a spherical wedge is equal to twice the angle of the lune which forms its base multiplied by the volume of a trirectangular pyramid.

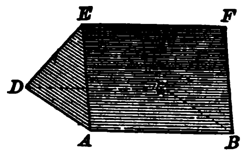
529. The volume of a spherical pyramid is equal to the spherical excess of its base multiplied by the volume of a trirectangular pyramid.

530. If the arcs of great circles be drawn from any point on a sphere to the extremities of another arc of a great circle, they will be greater than the sum of two other arcs similarly drawn but included by them.

SUPPLEMENT TO SOLID GEOMETRY.

THE WEDGE AND PRISMOID.

622. A **Wedge** is a solid bounded by a rectangle, two trapezoids, and two triangles.



The rectangle ($ABCD$) is the *back.*

The trapezoids ($ABFE$ and $DCFE$) are the *faces.*

The triangles (ADE and BCF) are the *ends.*

The line EF (where the faces meet) is the *edge.*

623. There are three distinct forms of the wedge:

1st. When the length of the edge is equal to the length of the back.

2d. When it is less.

3d. When it is greater.

624. Case I.—*When the length of the edge is equal to the length of the back.*

It is evident that this form of the wedge is a right triangular prism whose base is the triangle ADE and whose altitude is EF.

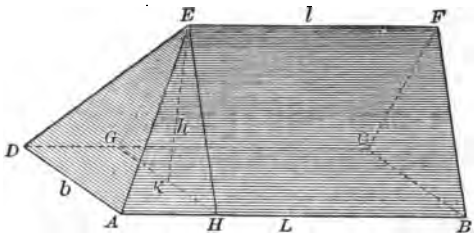
Hence its volume $= \triangle ADE \times EF$. § 464

EXERCISES.

531. Find the volume of a wedge whose back is 10 in. by 6 in., the altitude of the end being 12 in.

532. Find the volume of a wedge whose back is 20 ft. by 12 ft. and the end an equilateral triangle.

625. CASE II.—*When the length of the edge is less than the length of the back.*



Through F, the middle point of the edge, pass the plane FCB perpendicular to the back, dividing the wedge into two equal parts, only one of which is represented in the figure.

Through E pass the plane EGH parallel to FCB, forming the right triangular prism $GHBC—F$, and the quadrangular pyramid $AHGD—E$.

Denote AB by L, EF by l, AD by b, EK by h, and the volume by V.

Then the prism $GHBC-F = \frac{1}{2}\,bhl$, § 464

And the pyramid $AHGD-E = b(L-l)\frac{1}{3}h$. § 483

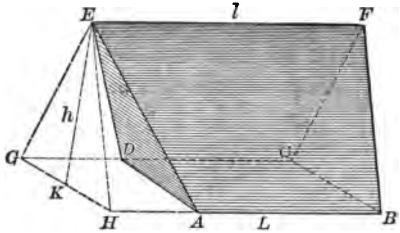
Hence $$V = \tfrac{1}{2}bhl + \tfrac{1}{3}\,bh(L-l),$$
$$= \tfrac{1}{2}bhl + \tfrac{1}{3}bhL - \tfrac{1}{3}bhl,$$
$$= \tfrac{1}{6}bhl + \tfrac{1}{3}bhL,$$

Or $$V = \tfrac{1}{6}bh(l + 2L).$$

By multiplying this expression by 2, letting l represent the entire length of the edge, and L the entire length of the back, we obtain the following

RULE.—*Add the length of the edge to twice the length of the back, and multiply this sum by one sixth of the product of the width and altitude; the final product will be the required volume.*

626. *When the length of the edge is greater than the length of the back.*



Through F, the middle point of the edge, pass the plane FCB perpendicular to the back, dividing the wedge into two equal parts, only one of which is represented in the figure.

Through E pass the plane EGH parallel to FCB, thus forming the right prism $GHBC-F$, and the quadrangular pyramid $AHGD-E$.

Denote AB by L, EF by l, AD by b, EK by h, and the volume by V.

Then the prism $GHBC-F = \frac{1}{2}bhl$, § 464

And the pyramid $AHGD-E = b(l-L)\frac{1}{3}h$. § 483

Hence $V = \frac{1}{2}bhl - \frac{1}{3}bh(l-L)$,

$= \frac{1}{2}bhl - \frac{1}{3}bhl + \frac{1}{3}bhL$,

$= \frac{1}{6}bhl + \frac{1}{3}bhL$,

Or $V = \frac{1}{6}bh(l + 2L)$.

By multiplying this expression by 2, letting l represent the entire length of the edge, and L the entire length of the back, we obtain the same rule as in § 625.

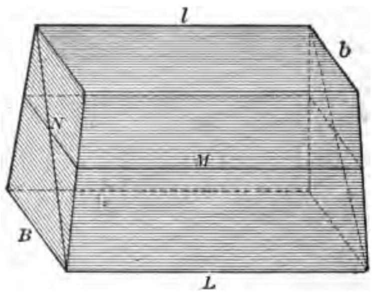
EXERCISE.

533. If the back of a wedge is 30 by 20 ft., the edge 24 ft., and the altitude 10 ft., what is the volume? Find the volume if the edge is 33 ft.

THE PRISMOID.

627. A **Prismoid** is the frustum of a wedge.

628. *To find the volume of a prismoid.*



Denote the length and width of the lower base by L and B respectively; the length and width of the upper base by l and b respectively; the length and width of the section equidistant from the two bases by M and N respectively; and the altitude by h.

Pass a plane through the edges l and L, dividing the prismoid into two wedges having the bases of the prismoid for backs, and the common altitude being the altitude of the prismoid.

The volume V of the prismoid is equal to the sum of the volumes of the two wedges.

Then $V = \frac{1}{6}Bh(l + 2L) + \frac{1}{6}bh(L + 2l)$, § 625

Or $V = \frac{1}{6}h(Bl + 2BL + bL + 2bl)$.

This may be written,

$$V = \frac{1}{6}h[BL + bl + (BL + Bl + bL + bl)]. \quad (1)$$

Since the section MN is equidistant from the bases,

Then $2M = L + l$,

And $2N = B + b$.

Multiplying these two equations,

$$4MN = BL + Bl + bL + bl.$$

Substituting this value in (1),

$$V = \tfrac{1}{6}h(BL + bl + 4MN).$$

But BL is the area of the lower base, bl is the area of the upper base, and MN is the area of the middle section. Hence, for finding the volume of a prismoid we obtain the following

RULE.—*Multiply the sum of the lower base, the upper base, and four times the middle section by one sixth of the altitude.*

It is readily shown that this rule will apply to finding the volume of a *pyramid, cone, prism, cylinder, frustum of a pyramid, frustum of a cone*, etc.

EXERCISES.

PRACTICAL EXAMPLES.

534. Find the volume of a rectangular prismoid, one of whose bases is 16 by 20 ft., the other 8 by 12 ft., and the altitude 24 ft.

535. How many bushels of lime in a kiln whose lower base is 10 by 12 ft., the upper base 16 by 18 ft., and the altitude 16 ft., if a bushel contains $1\frac{9}{16}$ cu. ft.?

536. What is the volume of a stick of hewn timber whose ends are 20 and 24 inches and 10 by 14 inches, its length being 30 ft.?

537. Find the volume of a square pyramid whose base is 6 ft. on a side and altitude 24 ft.

538. Find the volume of a cone whose altitude is 48 ft. and the radius of the base 6 ft.

539. Find the volume of the frustum of a triangular pyramid the sides of whose lower base are 8 in. and the sides of the upper base 4 in., its altitude being 18 in.

540. Find the volume of the frustum of a cone, the radius of the lower base being 16 in., the radius of the upper base 4 in., and the altitude 12 in.

541. Find the volume of a prismoid one of whose bases is 24 by 36 ft., the other 20 by 22 ft., its altitude being 4 rods.

MISCELLANEOUS PROPOSITIONS.

BOOK I.

542. If one angle of a triangle is equal to three times another angle of the triangle, the triangle can be divided into two isosceles triangles.

543. The sum of the three medial lines of a triangle is less than the perimeter and greater than the semi-perimeter.

544. If two of the medial lines of a triangle are equal, the triangle is isosceles.

545. The sum of the medial lines of a triangle is greater than three fourths of the perimeter of the triangle.

546. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

547. The bisectors of the angles of a trapezium form a second trapezium, the opposite angles of which are supplementary.

548. The lines joining the middle points of the sides of a square, taken in order, form a square.

549. If a diagonal of a quadrilateral bisects two angles, the quadrilateral has two pairs of equal sides.

550. In a right triangle a perpendicular let fall from the vertex of the right angle upon the hypotenuse cuts off two triangles mutually equiangular to the original triangle.

551. Prove that in a regular n-gon each angle equals $\frac{(n-2)180^\circ}{n}$.

552. What fractional part of each interior angle does each exterior angle of a regular n-gon equal?

553. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

554. In any triangle, the sides of the vertical angle being unequal, the median line drawn from the vertical angle lies between the bisectrix of the vertical angle and the longer side.

555. In any triangle, the sides of the vertical angle being unequal, its bisectrix lies between the median line and the perpendicular from the vertical angle to the base.

556. The triangle formed by joining points in the sides of an equilateral triangle equidistant from the vertices, taken in order, is an equilateral triangle.

557. The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

558. The line which joins points in the side of an isosceles triangle equally distant from the extremities of the base is parallel to the base.

559. The angle formed by drawing two lines from a point without a line to its extremities is less than the angle formed by two other lines similarly drawn but included by them.

560. The lines bisecting the two interior angles on the same side of a secant to two parallel straight lines meet at right angles.

561. The bisectors of the exterior angles of a triangle form a triangle whose angles are respectively the half sums of the angles of the given triangle taken two and two.

BOOK III.

562. The chords joining the corresponding extremities of two parallel chords are equal.

563. Intersecting chords that join the extremities of equal arcs are equal.

564. If two chords which intersect each other have one part of each equal, the two chords are also equal.

565. The tangents drawn through the vertices of an inscribed rectangle form a rhombus.

566. The bisectors of the angles contained by the opposite sides produced of an inscribed quadrilateral intersect at right angles.

567. If two straight lines are drawn through the point of contact of two circles tangent internally, the chords of the intercepted arcs are parallel.

568. If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two sides, the quadrilateral may be circumscribed about a circle.

569. The centre of a circle and the middle points of any arc and its chord are in one and the same straight line.

570. If AB is a common tangent to two circles which touch each other externally at C, prove that ACB is a right angle.

571. If a rectangle be inscribed in a circle, the tangents drawn through its vertices will form a rhombus.

572. If ACB is any angle at the centre of a circle, and if BE is drawn

meeting AC produced in E, and the circumference in D, so that DE shall be equal to the radius of the circle, then the angle E will be equal to one third of the angle ACB.

NOTE.—If BE could be drawn under these conditions with the straight line and the circumference only, ACB being any given angle, then the *trisection of an angle* in general would be possible by elementary geometry; but this has never been done.

573. If a circle is inscribed in a triangle, the distance from the vertex of any angle to the points of contact of its sides is equal to the semi-perimeter minus the side lying opposite to this angle.

574. If two circles are tangent to each other internally at the point B, and a chord AC of the larger is tangent to the smaller at D, then BD bisects the angle ABC.

575. If the opposite angles of a quadrilateral be supplementary, the quadrilateral can be inscribed in a circle.

576. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.

577. If two straight lines are drawn through the point of contact of two circles tangent externally, the chords of the intercepted arcs are parallel.

578. If AB is the common chord of two intersecting circles, and AC and AD are the diameters drawn from A, prove that the line CD passes through B.

579. The line joining the centre of a square described on the hypotenuse of a right triangle with the vertex of the right angle bisects the right angle of the triangle.

580. If two circles intersect each other, the longest common secant that can be drawn through either point of intersection is parallel to the line joining the centres of the circles.

581. Prove that the bisectors of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

CONSTRUCTIONS.

582. Draw two lines that shall divide a given right angle into three equal parts.

583. Draw a line tangent to a given circle and perpendicular to a given straight line.

584. Construct a circle of given radius tangent to a given circle and passing through a given point.

585. In a given isosceles triangle, draw a line that shall cut off a trapezoid whose base is the base of the given triangle and whose three other sides shall be equal to each other.

586. Through a given point draw a straight line making equal angles with two given straight lines.

587. Draw a circle that shall be tangent to a given circle and also to a given line at a given point.

588. In the prolongation of any diameter of a given circle find a point such that a tangent from it to the circumference shall be equal to the diameter of the circle.

589. Construct a parallelogram whose area and perimeter are respectively equal to the area and perimeter of a given triangle.

590. To find a point within a given triangle such that the three straight lines drawn from it to the vertices of the triangle shall make three equal angles with each other.

BOOK IV.

591. Any chord of a circle is a mean proportional between its projection on the diameter drawn from one of its extremities and the diameter itself.

592. If on a diameter of a circle two points are taken equally distant from the centre, the sum of the squares of the distances of any point of the circumference from these two points is constant.

593. The square described on the sum of two lines is equivalent to the sum of the squares described on the lines, increased by twice the rectangle of the lines.

594. The square described on the difference of two lines is equivalent to the sum of the squares described on the lines, diminished by twice the rectangle of the lines.

595. The rectangle contained by the sum and difference of two lines is equivalent to the difference of their squares.

596. If two fixed parallel tangents are cut by a variable tangent, the rectangle of the segments of the latter is constant.

597. The square of a side of a regular inscribed decagon together with the square of the radius equals the square of a side of a regular inscribed pentagon.

598. If two circles touch each other, secants drawn through their point of contact and terminating in the two circumferences are divided proportionally at the point of contact.

599. The rectangle of two sides of any triangle equals the rectangle of the perpendicular to the third side and the diameter of the circumscribed circle.

600. If two circles are tangent externally, the portion of their common tangent included between the points of contact is a mean proportional between the diameters of the circles.

601. In any isosceles triangle the square of one of the equal sides is equal to the square of any straight line drawn from the vertex to the base, plus the product of the segments of the base.

CONSTRUCTIONS.

602. Inscribe a square in a given triangle.

603. In a given triangle inscribe a rectangle whose sides are in a given ratio.

604. Construct a square equivalent to a hexagon.

605. Construct a triangle, having given one angle, the side opposite, and the area.

606. Divide a trapezoid into two equivalent parts by drawing a line parallel to its bases.

607. Construct a square, having given the difference between the diagonal of the square and its side.

BOOK V.

608. Prove that but three equal regular polygons may be used to cover a plane surface.

609. The area of a regular inscribed octagon is equal to that of a rectangle whose adjacent sides are equal to the sides of the inscribed and circumscribed squares.

610. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisectors of the opposite sides.

611. The rectangle of the segments of chords passing through a common point is a constant quantity.

612. The radius of any inscribed regular polygon is a mean proportional between its apothem and the radius of a similar circumscribed regular polygon.

613. Two diagonals of a regular pentagon not drawn from a common vertex divide each other in extreme and mean ratio.

614. If three circles intersect one another, their common chords will meet in the same point.

BOOKS VI., VII., AND VIII.

615. Find the locus of all points equally distant from the three edges of a trihedral angle.

616. Prove that if three lines are perpendicular to a fourth line at the same point, the first three lines are in the same plane.

617. Find the locus of the points in a given plane, the difference of the squares of the distances of which from two given points without the plane is constant.

618. Between two straight lines not in the same plane a common perpendicular can be drawn.

619. Any plane passing through the middle points of two opposite edges of a tetrahedron divides the tetrahedron into equivalent solids.

620. Any plane passed through the centre of a parallelopiped divides it into two equivalent solids.

621. Prove that the surface of a regular tetrahedron is equal to the square of a linear edge multiplied by the square root of 3.

622. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by one fourth the sum of the lateral edges.

623. The volumes of polyhedrons circumscribed about the same sphere are proportional to their surfaces.

624. The volume of a cylinder of revolution is equal to the area of the rectangle which generates it, multiplied by the circumference generated by the point of intersection of the diagonals of the rectangle.

625. Find the locus of all points, the distances of each one of which from two given spheres of equal radii are equal.

626. The volume generated by a triangle revolving about an axis in its plane and passing through its vertex without cutting its surface is equal to the area generated by its base multiplied by one third of its altitude.

FORMULÆ OF MENSURATION.

Abbreviations.—b = base; h = altitude; A = area; p = perimeter; r = radius; d = diameter; c = circumference; s = slant height; V = volume; m = middle section; e = apothem; a = arc; E = spherical excess; T = area of a tri-rectangular triangle.

The numbers refer to the page.

POLYGONS.

Rectangle, 148, $A = bh.$
Parallelogram, 149, $A = bh.$
Triangle, 150, $A = \frac{1}{2}bh.$
When a, b, and c represent the three sides and s their half sum, 181, $A = \sqrt{s(s-a)(s-b)(s-c)}.$
Trapezoid, 151, $A = \frac{1}{2}(b + b')h.$
Regular polygon, 211, $A = \frac{1}{2}pe.$

CIRCLE.

Circumference of a circle, 216, $c = \pi d = 2\pi r.$
Area of a circle, 216, 217, $A = \frac{1}{2}cr = \pi r^2 = \frac{1}{4}\pi d^2.$

POLYHEDRONS.

Prism, 284, $V = bh.$
Lateral area, 272, $A = ph.$
Parallelopiped, 282, $V = bh.$
Pyramid, 294, $V = \frac{1}{3}bh.$
Lateral area, 287, $A = \frac{1}{2}ps.$
Frustum of pyramid, 295, $V = \frac{1}{3}h(b + b' + \sqrt{bb'}).$
Lateral area of frustum, 287, $A = \frac{1}{2}(p + p')s.$
Prismoid, 373, $V = \frac{1}{6}h(b + b' + 4m).$

CYLINDER, CONE, AND SPHERE.

Right circular cylinder, 315, $V = bh = \pi r^2 h.$
Lateral area, 314, $A = ch = 2\pi rh.$
Right circular cone, 322, $V = \frac{1}{3}bh = \frac{1}{3}\pi r^2 h.$
Lateral area, 321, $A = \frac{1}{2}cs = rs\pi.$
Frustum of right circular cone, 326, $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr').$
Lateral area, 324, $A = s(r + r')\pi.$

Surface of sphere, 330,	$A = cd = 4\pi r^2 = \pi d^2$.
Surface of a lune, 363,	$A = 2aT$.
Surface of sph. triangle, 364,	$A = TE$.
Surface of sph. polygon, 365,	$A = TE$.
Surface of a zone, 331,	$A = 2\pi rh$.
Volume of sphere, 333,	$V = \frac{1}{3}Ar = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$.
Volume sph. sector, 334,	$V = \frac{2}{3}\pi r^2 h = \frac{1}{3}br$.
Volume sph. segment, 336,	$V = \frac{1}{2}\pi h(r^2 + r'^2) + \frac{1}{6}\pi h^3$.

EXPRESSIONS INVOLVING π.

$\pi = 3.141593$.	$\frac{4}{3}\pi = 4.188790$.	$\sqrt{\pi} = 1.772453$.
$\frac{1}{4}\pi = 0.785398$.	$\frac{1}{\pi} = 0.318309$.	$\frac{180^\circ}{\pi} = 57^\circ.29578$.

USEFUL ROOTS.

$\sqrt{2} = 1.4142$.	$\sqrt{10} = 3.1623$.	$\sqrt[3]{5} = 1.7100$.
$\sqrt{3} = 1.7321$.	$\sqrt{\frac{1}{2}} = 0.7071$.	$\sqrt[3]{6} = 1.8171$.
$\sqrt{5} = 2.2361$.	$\sqrt[3]{2} = 1.2599$.	$\sqrt[3]{7} = 1.9129$.
$\sqrt{6} = 2.4495$.	$\sqrt[3]{3} = 1.4422$.	$\sqrt[3]{9} = 2.0801$.
$\sqrt{7} = 2.6458$.	$\sqrt[3]{4} = 1.5874$.	$\sqrt[3]{10} = 2.1544$.

AREAS OF REGULAR POLYGONS WHOSE SIDE IS 1.

Name.	Sides.	Area.	Name.	Sides.	Area.
Triangle	3	0.433012.	Octagon	8	4.828427.
Square	4	1.000000.	Nonagon	9	6.181824.
Pentagon	5	1.720477.	Decagon	10	7.694208.
Hexagon	6	2.598076.	Undecagon	11	9.365639.
Heptagon	7	3.633912.	Dodecagon	12	11.196152.

VOLUMES OF REGULAR POLYHEDRONS WHOSE SIDE IS 1.

Name.	No. of Faces.	Volume.
Tetrahedron	4	0.117851.
Hexahedron	6	1.000000.
Octahedron	8	0.471404.
Dodecahedron	12	7.663118.
Icosahedron	20	2.181695.

ANSWERS TO NUMERICAL EXERCISES.

Page 21.—*Ex. 1.* 110°; *2.* 60°; *3.* 72°; *4.* 82°.

Page 25.—*Ex. 5.* 30°, 15°, −30°; *6.* 130°, 40°, −30°; *7.* $\frac{3}{5}$ of a rt. angle, $\frac{2}{7}$ of a rt. angle, $-\frac{2}{3}$ of a rt. angle; *8.* $\frac{6}{5}$ of a rt. angle, $\frac{2}{5}$ of a rt. angle, $-\frac{2}{5}$ of a rt. angle; *9.* 60°; *10.* 135°.

Page 27.—*Ex. 12.* 55°; *13.* 45°; *14.* 50° and 130°.

Page 39.—*Ex. 24.* 29° and 61°; *25.* 30° and 150°.

Page 52.—*Ex. 30.* 80°; *31.* 64°; *32.* 60°; *33.* 30°, 30°, 120°.

Page 55.—*Ex. 35.* 45°, 135°, 45°.

Page 63.—*Ex. 42.* $128\frac{4}{7}$°, 135°, 140°, 144°, $147\frac{3}{7}$°, 150°; *43.* 120°, 90°, 72°, 60°, $51\frac{3}{7}$°, 45°.

Page 69.—*Ex. 55.* 40°, 100°; *56.* 50°; *57.* 110°, 70°, 110.

Page 87.—*Ex. 105.* 6, $\sqrt{ab}$; *106.* 25; *107.* 30; *108.* $4:2 = x:y$; *109.* $6:8 = b:a$; *110.* $x:y = a:b$; *111.* $2y : x+y = 2b : a+b$; *112.* $a:c = 1:b$; *113.* 8 and 20.

Page 115.—*Ex. 145.* 60°; *146.* 25°; *147.* 80°, 120°, 160°; *148.* Angles 50° and 70°, arcs 120° and 140°; *149.* Angles 120° and 110°, arcs 80°, 100°, and 140°; *150.* 140° and 220°.

Page 117.—*Ex. 151.* 60°; *152.* 30°; *153.* 228°; *154.* 20°, 60°, and 100°.

Page 118.—*Ex. 155.* A = 80°, B = 120°, C = 100°, and D = 60°; *156.* D = 60°, E = 50°, and F = 70°.

Page 154.—*Ex. 224.* 3A. 32P.; *225.* 2A. 112P.; *226.* 10A.

Page 155.—*Ex. 227.* Alt. = 10.3926+ ft., area = 62.35+ sq. ft.; *228.* Perp. = 26.458+ in.

Page 163.—*Ex. 229.* 12, 9; *230.* 45, 15; *231.* $\frac{dc}{a}$, $\frac{bd}{a}$.

Page 164.—*Ex. 232.* 20 in.

Page 171.—*Ex. 233,* 4 to 9, 4 to 9.

Page 178.—*Ex. 241.* 72.111+ft.; *242.* 51.419+ft.; *243.* 81+ft.; *244.* 39.19+ sq. ft.; *245.* 57.3958+ ft.

Page 179.—*Ex. 246.* 12.393+ in.

Page 217.—*Ex. 294.* 4 to 9; *295.* 36π.

Page 223. *Ex. 341.* 25π in.; *342.* 286.4781+ sq. ft.; *343.* $D =$ 18 ft., Circ. $= 18\pi$; *344.* 28 in.; *345.* $5\sqrt{2}$; *346.* 4 in.; *347.* 16π sq. ft.; *348.* 6 in.; *349.* 288 sq. in.; *350.* 608.67744+ sq. in.; *351.* 166.2816+ sq. in.; *352.* 282.84+ sq. in.; *353.* 57°.29578+; *354.* 7.425+ in.

Page 261.—*Ex. 384.* 12 and 9.

Page 272.—*Ex. 400.* 600 sq. in.; *401.* 813.9514+ sq. in.

Page 275.—*Ex. 406.* 475.06+ sq. in.; *407.* 2389.85+ sq. ft.

Page 287.—*Ex. 411.* 200 sq. yd., 150 sq. yd.

Page 291.—*Ex. 412.* 62.3537+; *413.* 87.89+ sq ft.

Page 307.—*Ex. 414.* 24 sq. ft.; *415.* $19\frac{1}{3}$ sq. ft.; *416.* 4500 sq. in.; *417.* 886.2768+ sq. in.

Page 308.—*Ex. 418.* 693 sq. in.; *419.* 616 sq. ft.; *420.* 3325.537+ sq. ft.; *421.* 1184 cu. in.; *422.* Vol. 27 to 64, areas 9 to 16; *423.* 11.43+ ft.; *424.* Lat. area = 171.702+ sq. ft., entire area = 265.232+ sq. ft., vol. = 249.414+ cu. ft.; *425.* 360.54+ sq. ft., 456 cu. ft.; *426.* 384 cu. ft.; *427.* 9.52+ ft.; *428.* 60; *429.* 6 ft.; *430.* Lat. area = 627.4098+, vol. = 2268.06−; *431.* 36 sq. ft.

Page 334.—*Ex. 450.* 2116π sq. in.; *451.* 972π cu. in.; *452.* 432π cu. in.

Page 336.—*Ex. 453.* Lat. area $= 800\pi$ sq. in., vol. $= 4000\pi$ cu. in.; *454.* Lat. area $= 16\sqrt{37}\pi$ sq. ft., vol. $= 128\pi$ cu. ft.; *455.* 153π sq. ft.; *456.* 1216π cu. ft.; *457.* Surf. $= 1024\pi$ sq. in., vol. $= 5461\frac{1}{3}\pi$ cu. in.; *458.* 80π sq. in.; *459.* 900π cu. in.

Page 337.—*Ex. 460.* $661\frac{1}{3}\pi$ cu. in.; *461.* $474\frac{2}{3}\pi$ cu. in.; *462.* 332.5632+ cu. in.; *463.* 443.4176π cu. ft.; *464.* $9\frac{7}{8}$ in.; *465.* 6; *466.* 1350π sq. in.; *467.* $\frac{1}{2}$; *468.* 900π sq. ft.; *469.* $1333\frac{1}{3}\pi$ cu. ft.; *470.* 128π sq. in.; *471.* 80π sq. in., $266\frac{2}{3}\pi$ cu. ft.; *472.* Lat. surf. $= 216\pi$ sq. ft., entire surf. $= 324\pi$ sq. ft., vol. $= 648\pi$ cu. ft.; *473.* $682\frac{2}{3}\pi$ cu. in.; *474.* 432π cu. in.; *475.* Lat. area $= 168\pi$ sq. in., vol. $= 583.71+\pi$ cu. in.; *476.* Radius = 5 in., slant height = 5 in., lat. area $= 70\pi$ sq. in.

Page 356.—*Ex. 496.* 100°, 30°, 17°; *497.* 96°, 60°, 110°; *498.* 2 times the radius.

Page 363.—*Ex. 511.* 72π sq. in.; *512.* 768π sq. ft.; *513.* $10\frac{2}{3}\pi$ cu. in.

Page 371.—*Ex. 531.* 360 cu. in.; *532.* 1247.112+ cu. ft.

Page 373.—*Ex. 533.* 2800 cu. ft., 3200 cu. ft.

Page 375.—*Ex. 534.* 4736 cu. ft.; *535.* $2027\frac{13}{25}$ bu.; *536.* $61\frac{1}{3}$ cu. ft.; *537.* 288 cu. ft.; *538.* 576π cu. ft.; *539.* 290.9928+ cu. in.; *540.* 1344π cu. in.; *541.* 42416 cu. ft.

GLOSSARY.

THIS glossary contains the etymology and meaning of many of the common terms of Geometry. Only the words of greatest value to the student have been selected.

KEY.—L.= Latin, Gr. = Greek, dim. = diminutive.

Acute. [L. *acutus* = sharp.] Sharp. An *acute angle* is an angle less than a right angle.

Adjacent. [L. *ad* = to, + *jacere* = to lie.] Lying near to.

Altitude. [L. *altus* = high.] Height.

Analysis. [Gr. *ana* = again, + *luein* = to loosen.] The resolving of problems by reducing them to equations.

Angle. [L. *angulus* = an angle.] The amount of divergence of two lines which meet at a common point.

Antecedent. [L. *ante* = before, + *cedere* = to go.] The first term of a ratio.

Axiom. [Gr. *axioma* = that which is assumed.] A self-evident truth.

Bisect. [L. *bi* = two, + *sectus* = cut.] To divide into two equal parts.

Centre. [L. *centrum* = centre.] A point within a circle equally distant from every point of the circumference.

Chord. [Gr. *chorde* = a string.] A straight line joining the extremities of an arc.

Circle. [L. *circulus*, dim. of *circus* = circle; or Gr. *kirkos* = circle.] A plane figure bounded by a curved line every point of which is equally distant from a point within, called the centre.

Circumference. [L. *circum* = around, + *ferre* = to bear.] The bounding line of a circle.

Circumscribe. [L. *circum* = around, + *scribere* = to write, to draw.] To write or draw around.

Commensurable. [L. *com* = together, + *mensurare* = to measure.] Having a common measure.

Complement. [L. *complementum*, from *com* = with, + *plere* = to fill.] The difference between an arc or angle and 90° or a right angle.

Concave. [L. *con* = with, + *cavus* = hollow.] Hollow or curved.

Concentric. [L. *con* = together, + *centrum* = centre.] Having the same centre.

Cone. [Gr. *konos* = a cone, a peak.] A round body whose base is a circle, and whose convex surface tapers uniformly to a point called the vertex.

Consequent. [L. *con* = together, + *sequi* = to follow.] The second term of a ratio.

Constant. [L. *con* = together, + *stare* = to stand.] A quantity of fixed value.

Converse. [L. *con* = together, + *vertere* = to turn.] A proposition in which the hypothesis and the conclusion of the direct proposition are reversed.

Convex. [L. *convexus* = vaulted.] Rising into a spherical form.

Corollary. [L. *corollarium* = a gift, money paid for flowers, from *corolla*, dim. of *corona* = a crown.] A truth easily deduced from a proposition.

Curvilinear. [L. *curvus* = curved, + *linea* = line.] Bounded by curved lines.

Cylinder. [Gr. *kylindros*, from *kyliein* = to roll.] A round body with uniform diameter and parallel and equal bases.

Decagon. [Gr. *deka* = ten, + *gonia* = an angle.] A polygon of ten sides.

Degree. [L. *de* = down, + *gradus* = a step.] One three hundred and sixtieth part of a circumference.

Diagonal. [Gr. *dia* = through, + *gonia* = an angle.] A line of a polygon connecting two angles not consecutive.

Diameter. [Gr. *dia* = through, + *metron* = a measure.] A straight line drawn through the centre of a circle and terminating on both sides in the circumference.

Dihedral. [Gr. *di* = two, + *hedra* = a base.] Having two plane faces.

Dimension. [Gr. *di* = apart, + *metiri* = to measure.] A measure in a single line; as length, width, etc.

Dodecagon. [Gr. *do* = two, + *deka* = ten, + *gonia* = an angle.] A polygon of twelve sides.

Dodecahedron. [Gr. *do* = two, + *deka* = ten, + *hedra* = a base.] A regular polyhedron of twelve bases.

Enneagon. [Gr. *ennea* = nine, + *gonia* = an angle.] A polygon of nine sides. See *Nonagon.*

Equiangular. [L. *aequus* = equal, + *angulus* = an angle.] Having equal angles.

Equilateral. [L. *aequus* = equal, + *latus, lateris* = a side.] Having equal sides.

Equivalent. [L. *aequus* = equal, + *valere* = to be strong.] Having equal area.

Escribed. [L. *e* = out, + *scribere* = to write.] Drawn without.

Frustum. [L. *frustum* = a piece.] The part that remains after cutting the top off with a plane parallel to the base.

Geometry. [Gr. *ge* = the earth, + *metron* = a measure.] The science of extension.

Hemisphere. [Gr. *hemi* = half, + *sphaira* = a sphere.] Half a sphere.

Heptagon. [Gr. *hepta* = seven, + *gonia* = an angle.] A polygon of seven sides.

Hexagon. [Gr. *hexa* = six, + *gonia* = an angle.] A polygon of six sides.

Hexahedron. [Gr. *hexa* = six, + *hedra* = a base.] A polyhedron bounded by six planes. A cube is a hexahedron.

Homologous. [Gr. *homos* = the same, + *logos* = a discourse.] Corresponding in relative position and proportion.

Hypotenuse. [Gr. *hupo* = under, + *teinein* = to stretch.] The side opposite the right angle in a right triangle.

Hypothesis. [Gr. *hupothesis* = foundation.] That which is assumed in the statement of a theorem.

Icosahedron. [Gr. *eikosi* = twenty, + *hedra* = a base.] A polyhedron bounded by twenty planes.

Incommensurable. [L. *in* = not, + *com* = together, + *mensurare* = to measure.] Not having a common measure.

Infinity. [L. *in* = not, + *finitus* = bounded.] Without limit.

Inscribed. [L. *in* = in, + *scribere* = to write.] Drawn within, as a line, an angle, or a rectilinear figure, within a curvilinear figure.

Isosceles. [Gr. *isos* = equal, + *skelos* = leg.] Having two legs or sides equal.

Lemma. [Gr. *lemma* = a thing taken for granted.] An auxiliary proposition.

Locus. [L. *locus* = a place.] In general, a point, line, or surface which contains all points having a common property, and no others.

Lune. [L. *luna* = the moon.] A portion of the surface of a sphere included between two semi-circumferences of great circles.

Maximum. [L. *maximus* = greatest, pl. *maxima*, superlative of *magnus* = great.] The greatest quantity or value attainable in a given case.

Median. [L. *medius* = the middle.] Running through the middle.

Minimum. [L. *minimus* = least, pl. *minima*, superlative of *parvus* = little.] The least quantity or value attainable in a given case.

Mixtilinear. [L. *miscere* = to mix, + *linea* = a line.] Containing straight and curved lines.

Nonagon. [L. *nonus* = ninth, + Gr. *gonia* = an angle.] A polygon of nine sides. See *Enneagon.*

Obtuse. [L. *obtusus* = blunt, from *ob* = upon, + *tundo, tusum* = to strike.] Not sharp. An obtuse angle is one that is greater than a right angle.

Octagon. [Gr. *octa* = eight, + *gonia* = an angle.] A polygon of eight sides.

Octahedron. [Gr. *octa* = eight, + *hedra* = a base.] A polyhedron bounded by eight planes.

Parallel. [Gr. *parallelos* = parallel, from *para* = beside, + *allelon* = one another.] Extending in the same direction.

Parallelogram. [Gr. *parallelos* = parallel, + *gramma* = a line.] A quadrilateral whose opposite sides are parallel.

Parallelopiped. [Gr. *parallelelos* = parallel, + *epi* = upon, + *pedon* = the ground.] A prism whose base is a parallelogram.

Pentagon. [Gr. *pente* = five, + *gonia* = an angle.] A polygon of five sides.

Pentedecagon. [Gr. *pente* = five, + *deka* = ten, + *gonia* = an angle.] A polygon of fifteen sides.

Perimeter. [Gr. *peri* = around, + *metron* = a measure.] The distance around a figure.

Perpendicular. [L. *perpendiculum* = a plumb-line, from *per* = through, + *pendere* = to hang.] A line or plane falling at right angles upon another line or plane.

π. [Gr. π = pi.] The initial letter of the Gr. *periphereia* = circumference. Used to designate the ratio of the circumference of a circle to its diameter.

Polygon. [Gr. *polus* = many, + *gonia* = an angle.] A plane figure bounded by straight lines.

Polyhedron. [Gr. *polus* = many, + *hedra* = a base.] A solid bounded by planes.

Postulate. [L. *postulatum* = a demand.] To assume without proof.

Prism. [Gr. *prisma* = something sawed.] A solid whose bases are equal and parallel polygons.

Prismoid. [Gr. *prisma* = something sawed, + *eidos* = form.] A body that approaches the form of a prism.

Projection. [L. *pro* = forth, + *jacere* = to throw.] The act of throwing forward.

Pyramid. [Gr. *puramis* = a pyramid.] A solid bounded by a polygon and triangles having a common vertex.

Quadrant. [L. *quadrans* = a fourth part.] The quarter of a circle, or of the circumference of a circle.

Quadrilateral. [L. *quatuor* = four, + *latus, lateris* = a side.] A polygon of four sides.

Radius. [L. *radius* = a spoke of a wheel.] A line drawn from the centre of a circle to the circumference.

Ratio (rā′shi-o or rā′sho). [L. *ratus* = to reckon.] The relation which one quantity has to another of the same kind.

Reciprocal. [L. *reciprocus* = returning.] Mutually interchangeable. Unity divided by the quantity.

Rectangle. [L. *rectus* = right, + *angulus* = an angle.] A parallelogram whose angles are right angles.

Rectilinear. [L. *rectus* = right, + *linea* = a line.] A plane figure bounded by straight lines.

Rhombus. [Gr. *rhombus*, from *rhombein* = to turn round.] An oblique-angled equilateral parallelogram.

Scalene. [Gr. *skalenos* = unequal.] Unequal. A scalene triangle is one in which no two of the sides are equal.

Scholium. [Gr. *schole* = a learned discussion.] A remark upon one or more propositions.

Secant. [L. *secare* = to cut.] A line that cuts another.

Sector. [L. *secare* = to cut.] A part of a circle comprehended between two radii and the included arc.

Segment. [L. *secare* = to cut.] A part cut off from a figure by a line or plane.

Semicircle. [L. *semi* = half, + *circulus* = a circle.] The half of a circle.

Sphere. [Gr. *sphaira* = a ball.] A solid bounded by a curved surface every point of which is equally distant from a point within called the centre.

Subtend. [L. *sub* = under, + *tendere* = to stretch.] To extend under, as the chord which subtends an arc.

Superposition. [L. *super* = over, + *ponere* = to place.] The act of superposing; being placed upon something.

Supplement. [L. *sub* = under, + *plere* = to fill.] The difference between an arc or angle and 180° or two right angles.

Synthesis. [Gr. *sun* = with, + *tithenai* = to place.] A putting together.

Tangent. [L. *tangere* = to touch.] A right line which touches a curve.

Tetrahedron. [Gr. *tetra* = four, + *hedra* = a base.] A polyhedron bounded by four bases.

Transversal. [L. *trans* = across, + *vertere* = to turn.] A line which intersects any system of other lines.

Trapezium. [Gr. *trapezia* = a table, from *tetra* = four, + *pous* = foot.] A quadrilateral none of whose sides are parallel.

Trapezoid. [Gr. *trapezia* = a table, + *eidos* = form.] A quadrilateral two of whose sides are parallel.

Trigon. [Gr. *tri* = three, + *gonia* = an angle.] A triangle.

Truncate. [L. *truncare* = to cut off.] To cut off; as a truncated cone.

Undecagon. [L. *undecim* = eleven, + Gr. *gonia* = an angle.] A polygon of eleven sides.

Zone. [Gr. *zone* = a girdle.] A portion of the surface of a sphere included between two parallel planes.

BIOGRAPHICAL NOTES.

In these notes will be found brief sketches of eminent mathematicians who have made valuable contributions to Geometry.

Anaxagoras (an-aks-ag′o-ras). Born about 500 B. C.; died about 428 B. C. A Greek philosopher and mathematician, for a long time resident in Athens, where he became the teacher of Pericles. He was imprisoned for his opinions relating to Astronomy, and spent much of his time in trying to square the circle.

Apollonius (ap-o-lo′ni-us). Born at Perga about 250 B. C.; died about 205 B. C. Next to Archimedes, he was the most illustrious of the ancient Greek geometricians. So great were his labors and genius that he was called the *Great Geometer*. His principal work was a treatise on *Conic Sections*, in eight books. He is said to have named three of these higher curves, the *parabola*, the *ellipse*, and the *hyperbola*.

Archimedes (är-ki-mē′dēz). Born at Syracuse about 287 B. C.; died at Syracuse about 212 B. C. The most celebrated geometrician of antiquity. His most important services were rendered to pure Geometry, but his fame rests chiefly on the application of mathematics to mechanics. He discovered the beautiful theorem that the sphere is two thirds of the circumscribing cylinder, and that their surfaces bear the same relation. (See Ex. 483 and 484, Book VIII.) He was the first to assign an approximate value to π. (See § 351.) Theorem XLI., Book I., is due to Archimedes.

Descartes (dā-kärt′), **René**. Born at La Haye, France, 1596; died at Stockholm, 1650. A celebrated French philosopher and mathematician. The inventor of what is now known as the *Cartesian* or *Analytical Geometry*.

Diophantus (di-ō-fan′tus). A Greek mathematician who

lived at Alexandria, probably in the 4th century A. D. The reputed inventor of *Algebra.*

Euclid (ū′klid). Lived at Alexandria about 300 B. C. An eminent writer on Geometry. His principal work is the *Elements of Geometry*, in thirteen books, parts of which have been used as a text-book for elementary geometry down to the present time. It is said of him that when Ptolemy asked if there was not some easier way of learning Geometry, he replied, "*There is no royal road to Geometry.*" Theorem VIII., Book IV., is the 47th proposition of the first book of Euclid's *Elements.* See Note, § 236.

Euler (oi′ler), **Leonhard.** Born 1707; died 1783. A Swiss scholar, and one of the most eminent mathematicians of modern times. Theorem XII., Book IV., is due to Euler.

Hippocrates (hip-pok′ra-tēz). Born about 460 B. C.; died about 377 B. C. An eminent Greek mathematician. The first to effect the quadrature of a curvilinear figure by finding a rectilinear one equal to it. He was also the author of the first elementary text-book on Geometry. Exercise 344 is due to Hippocrates.

Kepler (kep′ler), **Johann.** Born 1571; died 1630. A German mathematician and physicist, celebrated for his work on Astronomy. He was the first to introduce the idea of infinity into the treatment of Geometry. He regards the circle as composed of an infinite number of triangles, the cone as composed of an infinite number of pyramids, etc.

Legendre (le-zhondr′), **Adrien Marie.** Born at Toulouse, 1752; died at Paris, 1833. A celebrated French mathematician. The author of the most celebrated work on Geometry written during the last century. It is called *Eléments de Géometrie*, and was published in 1794.

Leibnitz (līb′nits), **Gottfried Wilhelm.** Born 1646; died 1716. A celebrated German philosopher and mathematician. The founder of the *Infinitesimal Calculus.*

Newton, Sir Isaac. Born at Woolsthorpe, 1642; died at Kensington, 1727. A famous English mathematician and natural philosopher. In 1664 he discovered the *Binomial Theorem*, and, about 1665, the *Differential Calculus.* His greatest work is the *Principia.*

Pappus (pap′us). Lived about the close of the 4th century. An Alexandrian geometer. His chief mathematical work was called *The Collection.*

Pascal (pas′kal), **Blaise.** Born at Clermont-Ferrand, 1623; died at Paris, 1662. A celebrated French geometrician, philosopher, and physicist. At the age of sixteen he achieved renown by his *Treatise on Conic Sections.* Later in life he undertook and solved many of the most difficult problems of mathematics.

Plato (plā′to). Born at Ægina, 429 or 427 B. C.; died at Athens, 347 B. C. An eminent Greek philosopher and mathematician. He made mathematics the basis of his teaching. *Conic Sections* were first studied in his school. He gave a simple and elegant method for the *duplication of the cube* and for the *trisection of an angle.* Plato was the first to limit Plane Geometry to the use of the *Straight Line* and the *Circumference.*

Ptolemy (tol′e-mi), **Claudius Ptolemæus.** Born 87 A. D.; died 165 A. D. One of the greatest astronomers, geographers, and mathematicians of the late Greeks. The Ptolemaic system of astronomy, which places the earth at the centre of the universe, is named in his honor. In Geometry Ptolemy has been ranked fourth among the ancients, after Euclid, Apollonius, and Archimedes. He fixed the approximate value of π at $3\frac{17}{120}$, or, in decimals, 3.1416. Theorem XXV., Book IV., is due to him.

Pythagoras (pi-thag′o-ras). Born about 582 B. C.; died about 500 B. C. A famous Greek philosopher and mathematician. He is supposed to have discovered the following theorems: *only three plane figures can fill up the space around a point; the sum of the angles of a plane triangle equals two right angles* (See Theorem XVII., Book I.); *the circle is greater than any other plane figure of equal area* (See § 360); and the celebrated proposition of the *square on the hypotenuse* (See Theorem VIII., Book IV.).

Thales (thā′lēz). Born at Miletus about 640 B. C.; died about 546 B. C. A famous Greek philosopher, astronomer, and geometer; one of the seven wise men of Greece. He introduced Geometry into Greece from Egypt. He is said to have determined the distance of vessels from the shore by Geometry; measured the height of the pyramids by means of their shadows; and discovered that all angles in a semicircle are right angles (See § 180). Theorem I., Book III., is due to Thales.

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